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APOLLO TELEMETRY FRAME SYNCHRONIZATION TECHNIQUES

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THE JOHNS HOPKINS UNIVERSITY • APPLIED PHYSICS LABORATORY

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THE JOHNS HOPKINS UNIVERSITY • APPLIED PHYSICS LABORATORY
8621 Georgia Avenue, Silver Spring, Maryland 20910

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THE JOHNS HOPKINS UNIVERSITY
APPLIED PHYSICS LABORATORY
SILVER SPRING, MARYLAND

ABSTRACT

An analysis has been made of data frame synchronization control for Apollo downlink telemetry. The basic objective was to determine the effects of varying the switch-controlled error tolerance thresholds of two types of existing operational equipment. Criteria for setting these error tolerance switches are presented. The criteria are established to control the average times spent in the acquisition and lock modes of the data decommutation process. Any choice of switch settings would be dependent upon the data quality and quantity requirements of a particular mission. Examples of switch setting criteria are therefore given which depend upon "maximum acceptable" bit error rates based on assumed operational requirements.

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INTRODUCTION

A PCM decommutator operates to decommutate or segment a consecutive series of binary pulse coded signals in accordance with a predetermined telemetry frame format. The format contains a reference mark or frame synchronization pattern. The decommutator must locate this pattern and verify that it is a true or valid pattern. Decommutation is then accomplished by referencing each data segment to the frame synchronization pattern.

The part of the PCM decommutator that performs the synchronization function is the synchronization subsystem. This subsystem has various preset constants that allow the synchronization threshold to be varied.

The purpose of this report is to provide insight into the effects of varying the frame synchronization threshold on (1) the average times associated with acquiring frame synchronization and (2) the probabilities of maintaining this synchronization over a given time interval. Synchronization thresholds are determined by allowing a preset number of bit errors in each synchronization pattern and a preset number of consecutive patterns to have bit errors exceeding the allowed number prior to reverting to an inferior synchronization mode. The threshold values are therefore presented in terms of switch settings on the PCM decommutators which provide practical application of the computation results.

In order to provide a synchronization method which indicates desirable threshold settings it is necessary to have a synchronization criterion. The criterion selected should suit the needs of the data user. Many criteria are available. Discussion of some criteria is provided to allow selection of one believed best for the particular use of the decommutators in the NASA Manned Space Flight Tracking Network.

This report is restricted to considering two PCM Decommutator Systems - the Dynatronics, Inc., (MSFTP-2), PCM Decommutator, and the Electro-Mechanical Research, Inc., Manned Space Flight Telemetry PCM System 1 (MSFTP-1). Both systems have similar frame synchronization subsystems with the exception that the Dynatronics System has an additional search phase not implemented in the MSFTP-1 System. Section 1 describes the operation of these subsystems.

Computation examples are presented using the Apollo Command Service Module (CSM) 51,200 bits per second telemetry format. This format consists of a 26 bit frame synchronization pattern immediately followed by a 6 bit frame identification (ID) code. A data frame is 1024 binary bits in length providing a frame rate,hence synchronization pattern rate, of 50 frames per second. One data frame therefore contains a synchronization pattern, an ID code, and 992 bits which convey information in 124 quanta or words of 8 binary bits.

All frame synchronization considerations in this report assume perfect bit clock regeneration. The clock regeneration is performed by a bit synchronizer subsystem. Received data are first processed through this subsystem of the decommutator which phase locks a regenerated bit clock to the incoming data. It then uses this clock to time and assist in squaring the non-return-to-zero data binary pulses. The bit clock regeneration or bit synchronization signal provides the basic clock for the PCM decommutator.

Formulas utilized in computing probabilities and average times are presented or developed in the Appendices. It will be noted that some formulas approximate the desired values. These approximations should be valid in general under the presented assumptions and do provide considerable computational simplification. Additionally, examples are shown of the probability distributions from which the average times were determined.

1. DESCRIPTION OF FRAME SYNCHRONIZATION SUBSYSTEM OPERATION

The frame synchronization subsystem operates in three sequential modes: search mode, check or verify mode, (called check mode for this report), and lock mode. Search mode operation examines the incoming serial data in 26 bit sets (for Apollo data), shifting out one old bit and shifting in one new bit, until a synchronization pattern is found. Finding the first pattern which can contain at most $E_{\emptyset_1}$ bit errors terminates the search mode for the MSFTP-1 System.

The Dynatronics System has a two-phase search mode. The first phase is the same as for the MSFTP-1 System. The second phase continues the search on a bit by bit basis for an additional time equal to one telemetry frame length less the number of bits in the synchronization pattern. If a pattern containing up to $E_{\emptyset_2}$ bit errors is found within this time the second phase terminates, terminating the search mode. If an additional pattern is not found, phase two terminates at the end of the frame length indicated by the first synchronization pattern. The value of $E_{\emptyset_2}$ is generally set to be less than the value of $E_{\emptyset_1}$. The two-phase search mode can increase the probability of terminating search operation with a valid frame synchronization pattern, as is shown later.

Search mode termination initiates the check mode. The check mode checks or verifies that a valid synchronization pattern occurs at the known frame length. A valid pattern is defined as one containing at most E_c bit differences from the true pattern. Additionally, the check mode can require F_c consecutive checks of the synchronization pattern. If a pattern contains more than E_c bit differences from the true pattern during the F_c times the pattern is checked, the synchronization subsystem returns to the search mode. If the F_c consecutive checks are satisfactory the check mode is terminated, initiating the lock mode.

The search and check modes are referred to as the frame synchronization acquisition phase. Completion of this phase in a minimum time is desirable to increase the time spent in lock mode operation. However, the amount of time spent in the acquisition phase is proportional to the probability of entering lock mode operation with valid frame synchronization. The amount of time is directly related to the number of times, $F_c + 1$, that a valid pattern is observed which decreases the probability that random combinations of bits would pass as valid synchronization patterns. Therefore, one compromise in establishing an operational frame synchronization criterion is the relation between minimizing acquisition time and maximizing the probability of valid synchronization.

The highest order or level of frame synchronization occurs when the synchronization subsystem operates in the lock mode. The synchronization pattern is examined each frame for the occurrence of a valid pattern. A valid pattern is defined as one containing up to E_L bit differences from the true pattern. Lock mode operation is instrumented such that F_L consecutive invalid patterns must be detected to cause loss of lock, automatically switching the synchronization subsystem to search mode operation.

The probability of maintaining operation in the lock mode is proportional to the number of allowed bit differences, E_L, and consecutive invalid frames, F_L. One operational criterion is to maximize the probability of maintaining synchronization lock. Since data output from the decommutator is generally programmed to occur only when the greatest probability of valid synchronization is available, maximizing the lock time would provide the greatest amount of output data.

The frame synchronization thresholds are set by the values of $E_{\emptyset_1}$, $E_{\emptyset_2}$, for the search mode; E_c, F_c, for the check mode, and $E_L F_L$, for the lock mode. Each of these values is instrumented by a "Digiswitch" setting on the appropriate PCM decommutator. The E switches allow synchronization pattern bit differences from the true pattern to be preset within the range of zero to nine bits. Each pattern satisfying the preset requirement is defined as a valid synchronization pattern.

The F switches allow a preset number of consecutive patterns within the range of one to nineteen to be established for synchronization subsystem operation. The setting F_c requires that this number of valid patterns be recognized in the check mode before advancing to the lock mode. If one invalid pattern is detected before F_c valid patterns are detected, synchronization operation reverts to the search mode. The setting of F_L requires that this number of invalid patterns must be detected before the lock level of synchronization is lost. The subsystem again reverts to search operation. If a series of invalid patterns numbering less than F_L occurs but one valid pattern is then detected, the count of invalid patterns is reset to zero. The subsystem remains in the lock mode.

Figures 1 and 2 are flow diagrams for the frame synchronization subsystem logic of both PCM decommutators considered. Figure 3 shows examples of synchronization operation.

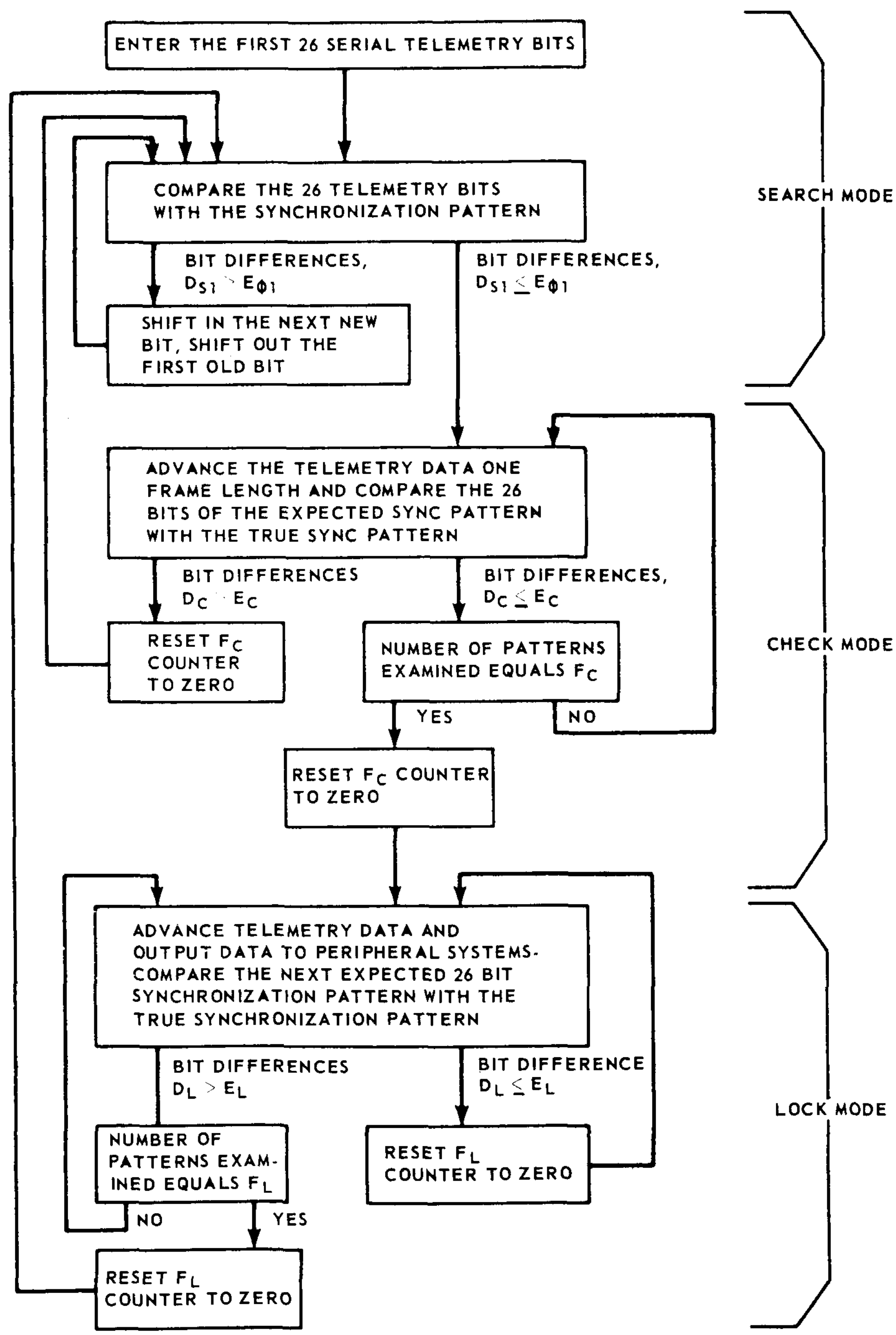


FIG. 1 FLOW DIAGRAM OF THE FRAME SYNCHRONIZATION LOGIC FOR THE MSFPT-1 SYSTEM

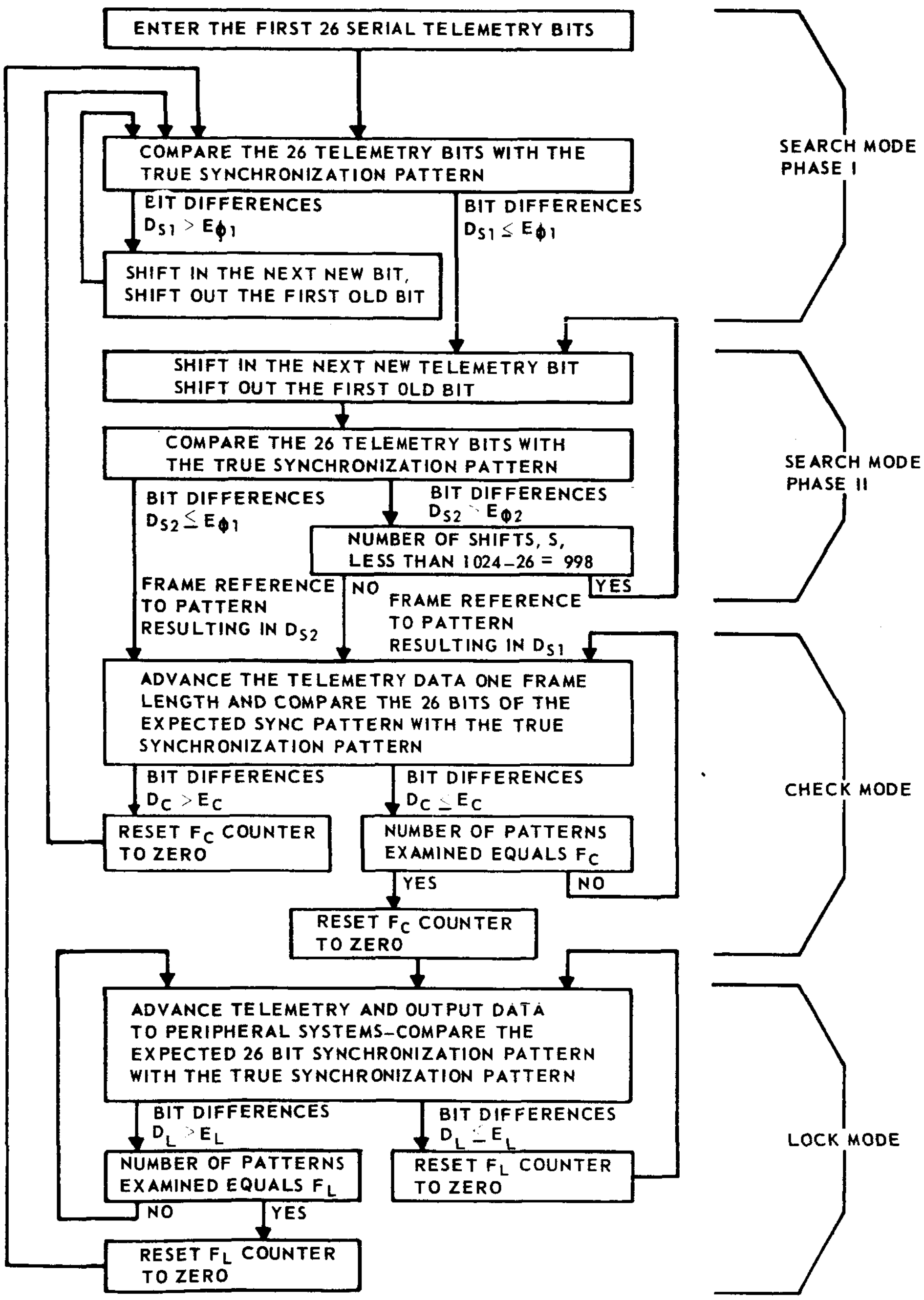
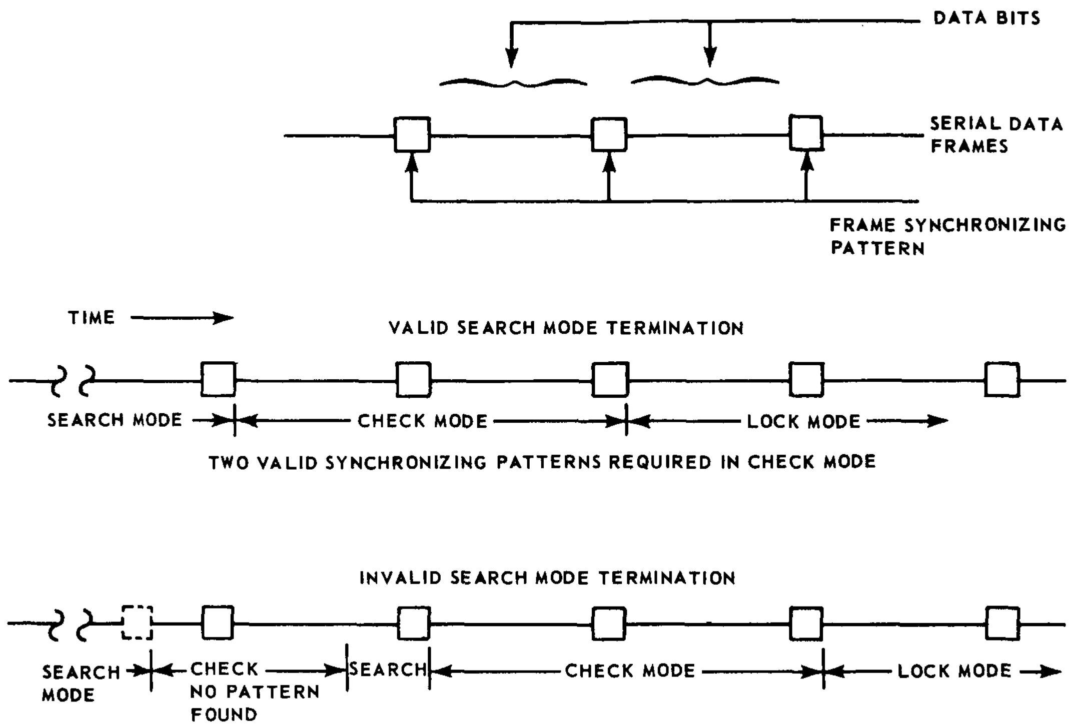
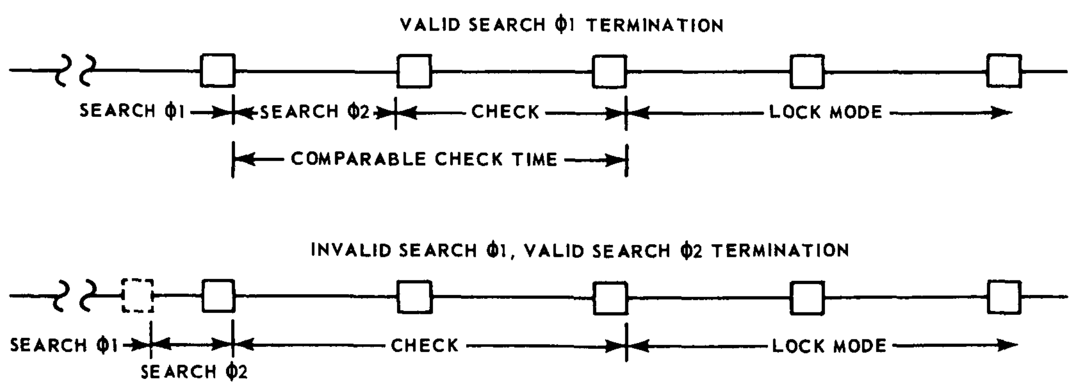


FIG. 2 FLOW DIAGRAM OF THE FRAME SYCHRONIZATION LOGIC FOR THE DYNATRONICS SYSTEM



(a) FRAME SYNCHRONIZATION FOR THE MSFTP-1 SYSTEM



(b) FRAME SYNCHRONIZATION FOR THE DYNATRONICS SYSTEM

FIG. 3 EXAMPLES OF SYNCHRONIZATION OPERATION

2. SYNCHRONIZATION CRITERIA

The first step in establishing synchronization thresholds for the PCM decommutators is to select a synchronization criterion suited to the particular user. The primary purpose of the decommutators is to provide synchronized data to peripheral equipment. This is generally programmed to occur when the synchronization subsystem is in lock mode operation. One criterion is then to set the E and F switches to maximize the time spent in this mode of operation. This criterion would maximize the amount of data output at the expense of a weighted (reduced) probability of valid frame synchronization and an average data quality controlled only by the quality of the input data.

Another criterion is to maximize the expected lock time if the data quality, as indicated by the quality of the synchronization pattern, is greater than some desired minimum value. This criterion would seek to maximize the probability of maintaining synchronization lock for a given maximum bit error rate (BER) and minimize the lock probability for data having a greater BER (poorer quality).

Many additional criteria could be listed; for example, a compromise between the two criteria stated, or minimizing the probability of obtaining false lock over a long time period, etc.

One possible disadvantage in maximizing the amount of data output is that data would be implicitly defined as all telemetry bits having a BER less than 0.5. Since we choose to define noise as having an expected BER of 0.5 [signal to noise ratio (SNR) equal to zero] information would then be available for use if the BER decreased to any value below 0.5 implying a SNR greater than zero. A practical limitation to the maximum BER that could be handled, however, is the ability of the bit synchronizer to acquire and maintain bit synchronization. Bit synchronization is obtained from the telemetry data stream. As such, a SNR greater than about 0.4 is required[1], indicating that a practical maximum BER less than about 0.3 could be handled with presently available bit synchronizers.

Additional considerations to enable a practical system to effectively operate at a BER of 0.3 would be the length of the frame synchronization pattern, the construction of the pattern (its "1" and "0" configuration) and the range of settings available on the PCM decommutator. These considerations would allow synchronization bit and pattern errors required to insure proper data synchronization and a reasonable probability of maintaining this synchronization lock. Since the main purpose of this paper is not to design a theoretical system but to establish synchronization methods for available systems, discussion of the above considerations is deferred. However, many References are available on the subject- 2, 3, 4, 5.

Since the PCM decommutator for which a synchronization criterion is desired is part of a real-time information system, maximizing the available data would at times present data with a greater BER than could be practically used. Most information is contained in eight bit binary words. Reducing the probability of receiving a zero error word could handicap the system since it would present large amounts of misleading and confusing information. Since redundancy coding and data editing are not presently performed it seems logical to require a high probability of an error free word.

A limited time would be available to process data even if processing were performed since the system must operate with a minimum data delay. Limiting the probability of data errors prior to processing would enable simpler and less time consuming computer processing procedures to be used.

Under the present system constraints it seems highly desirable to limit the maximum BER for PCM decommutator output data. This maximum value may be different for possible future systems, however.

Therefore, the criterion to maximize the probability of maintaining synchronization lock with a particular maximum acceptable BER and minimizing lock probability if the data have a greater BER would appear to be best for the present information system. This criterion requires the maximum BER to be established before proceeding with the synchronization method.

A practical way to establish a maximum BER is to assume that the bit errors are independent. The probability of an eight bit word having k bits in error is where q_b is the BER, $p_b = 1-q_b$, and $\binom{8}{k}$ represents the binomial coefficients. This relation is shown as Eq (1).

$$p(k) = \binom{8}{k} q_b^{\,k}\, p_b^{\,8-k} \qquad \begin{array}{l}\text{BER} = q_b \text{ is used by}\\ \text{convention; } q_b \text{ is independent}\\ \text{at the bit rate}\end{array} \qquad (1)$$

The probability that any eight bit word is received error free, $k = 0$ in Eq. (1), is then 0.430, 0.923, and 0.992 when the BER is 1×10^{-1}, 1×10^{-2}, and 1×10^{-3}, respectively. The probability of a valid bit, indicating an event, would be 0.9, 0.99, and 0.999 with the same BER's as stated.

It appears that an acceptable BER would be on the order of 1×10^{-2} resulting in a probability of better than 90 percent that a valid data word is received. A better maximum BER would be 1×10^{-3} increasing this probability to more than 99 percent. Circuit margin studies have indicated that the BER will in general be less than 1×10^{-3} for the majority of the time that data are being received.[6,7] Therefore, either maximum BER would be a reasonable value to select, assuring that a large amount of information would be supplied to the system with a satisfactory maximum average error rate. It could then be assumed that data were not available or of poor quality if new data were not supplied for system use.

The circuit margin equations[8] are generally set to require a SNR providing a BER of 1×10^{-6}. If a maximum BER of 1×10^{-2} is acceptable, a SNR about 6 db less than required for 1×10^{-6} BER data will allow successful telemetry information decommutated. A maximum BER of 1×10^{-3} would allow a 4 db SNR reduction below the value necessary for 1×10^{-6} BER data.

Therefore, the criterion to establish a synchronization method for this paper will be to maximize the probability of data output with the constraint of a maximum acceptable BER either 1×10^{-2} or 1×10^{-3}, and to minimize the probability of lock with poorer quality data.

3. ACQUISITION PHASE SYNCHRONIZATION METHODS

The frame synchronization subsystem, during the acquisition phase, functions to search the incoming serial telemetry data to locate the first bit pattern that differs from the true frame synchronization pattern by no more than $E_{\emptyset 1}$ bits. The first pattern is then checked by comparing a second pattern referenced one telemetry data frame length from the first pattern. This second pattern can differ by no more than E_c bits from the true pattern. If the two patterns meet the preset bit difference settings a minimum time is spent in the search and check modes. The synchronization subsystem then advances to lock mode operation, and the PCM decommutator begins to supply synchronized data into the information system.

The probability of valid frame synchronization preceeding operation in the lock mode may be increased by requiring additional pattern comparisons in the check mode. The decommutator can be set to require F_c valid pattern detections, each one frame length apart, before switching to the lock mode.

Figure 4 shows the probabilities of leaving the three synchronization modes. Capital P refers to the probability of successfully terminating the search mode with detection of a valid synchronization pattern, and Q = 1-P. The value p_c is the probability of successfully detecting a valid frame pattern in the check mode. Since F_c frames may be required, p_c raised to the r ($r = F_c$) power is the probability of successfully detecting F_c consecutive frame patterns and advancing to the lock mode. Hence, $1 - p_c^r$ is the probability of returning operation to the search mode rather than advancing to the lock mode. The same symbols with the additional subscript "f" define the probabilities of leaving the check mode after an invalid or false synchronization entry. The value t_n is the probability that a failure run will not terminate successful operation in the lock mode before some number "n" of frame pattern detections. Therefore, $1 - t_n$ is the probability of terminating successful lock operation before n frames have been examined. Similarly, $1 - t_n'$ is the probability of leaving the lock mode after entering it with invalid frame synchronization.

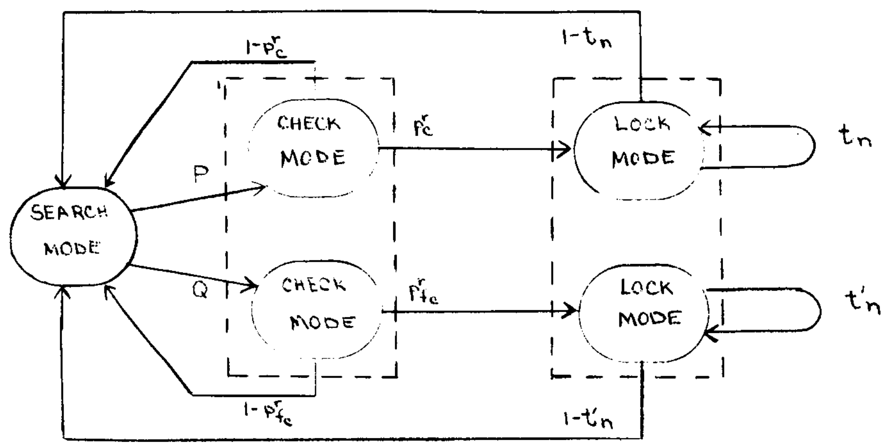


FIG. 4 PROBABILITIES OF LEAVING THE THREE SYNCHRONIZATION MODES

P = Probability of terminating search mode with valid synchronization

Q = Probability of terminating search mode with invalid synchronization, Q = 1-P.

p_c^r = Probability that each of the first r consecutive synchronization patterns are detected having up to an allowed number of bits different from the true pattern (r patterns required to enter the lock mode).

$1-p_c^r$ = Probability that at least one of the first r synchronization patterns has more than an allowed number of bits different from the true pattern.

$p_{f_c}^r$ = Probability that r random bit patterns appear as synchronization patterns

$1-p_{f_c}^r$ = Probability that at least one random pattern has more than an allowed number of bit differences from the true pattern.

t_n = Probability of maintaining lock mode operation for some time t.

$1-t_n$ = Probability of returning to the search mode before some time t is spent in lock mode operation.

t'_n = Probability of random patterns appearing as synchronization patterns causing maintenance of lock mode operation for some time t'.

$1-t'_n$ = Probability of returning to search mode before some time t' is spent in lock mode operation.

The synchronization criterion requires that the probabilities, P, p'_c and t_n be as large as possible for data with an acceptable BER while keeping these values as small as possible for poorer quality data. However, for any quality data it is desirable to keep Q, p_{f_c}, and t'_n small. The false synchronization probabilities could conceivably allow the decommutators to stay in a false synchronization loop. False synchronization is indicated by operation in the lower part of Figure 4.

The probability, P, of terminating the search mode with detection of a valid frame synchronization pattern is determined from Eq(2). This Equation derived in Appendix I is applicable to the Dynatronics two-phase search mode. It is applicable to the MSFTP-1 decommutator by setting the part within brackets to unity.

$$P = \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} \left[q_{f_2}{}^{m-1} + \frac{p_{s_2}\, p_{f_1}}{p_{s_1}\, p_{f_2}} \left(1 - q_{f_2}{}^{m-1}\right) \right] \quad (2)$$

p_{s_1} = Probability of valid pattern in phase 1 as functions of BER and $E_{\emptyset_1}$

p_{s_2} = Probability of valid pattern in phase 2 as functions of BER and $E_{\emptyset_2}$

p_{f_1} = Probability of an invalid pattern detection as a function of $E_{\emptyset_1}$

p_{f_2} = Probability of an invalid pattern detection as a function of $E_{\emptyset_2}$

$q_{f_2} = 1 - p_{f_2}$

m = Approximate number of uncorrelated patterns in one telemetry frame of 1024 bits of which only one would be a valid synchronization pattern.

The probability of having random noise or data bits appear as a frame synchronization pattern is determined from Eq (3).

$$p_f \; (k \leq E) = 2^{-26} \sum_{k=0}^{E} \binom{26}{k} \qquad (3)$$

The probability p_f takes on the subscripts for the particular phase or mode considered. The value E is the allowed number of bit disagreements between the detected and true synchronization pattern; E also takes on the subscript for the particular phase or mode considered. It is assumed for this case that the probability of a bit error q_b is 0.5.

The telemetry demodulator will provide true or complement data to the decommutator. Therefore, the decommutator is programmed to invert all data if a complement synchronization pattern is detected. This automatic data inversion allows either a true or complement set of random bits to be recognized as a synchronization pattern. For the Apollo system considered, the values of Eq(3) are multiplied by 2 and E is restricted to values less than 13.

Values computed from Eq(3) are shown in Table I for a range of E from 0 to 5 bits. Also the values of p_f^r are shown with r varying from 1 to 5. This table may be used, for example, to find the probability of leaving the check mode if E is set to 2 bits and $F_c = r$ is set to 3 frames as $p_{f_c}^r = 1.2 \times 10^{-15}$. Values of p_{f_1} and $q_{f_2} = 1-p_{f_2}$ are taken from the r = 1 row and column indicating $E_{\emptyset_1}$ or $E_{\emptyset_2}$ bits.

TABLE I

Probabilities of random noise or data agreeing with the 26 bit true or complement synchronization pattern r times when E errors are allowed within each pattern.

r	E = 0	E = 1	E = 2	E = 3	E = 4	E = 5
1	$3.0x10^{-8}$	$8.0x10^{-7}$	$1.0x10^{-5}$	$8.8x10^{-5}$	$5.3x10^{-4}$	$2.5x10^{-3}$
2	$8.9x10^{-16}$	$6.5x10^{-13}$	$1.1x10^{-10}$	$7.7x10^{-9}$	$2.8x10^{-7}$	$6.2x10^{-6}$
3	$2.6x10^{-23}$	$5.2x10^{-19}$	$1.2x10^{-15}$	$6.8x10^{-13}$	$1.5x10^{-10}$	$1.6x10^{-8}$
4	$7.9x10^{-31}$	$4.2x10^{-25}$	$1.2x10^{-20}$	$6.0x10^{-17}$	$8.1x10^{-14}$	$3.9x10^{-11}$
5	$2.4x10^{-38}$	$3.4x10^{-25}$	$1.2x10^{-20}$	$6.0x10^{-17}$	$4.3x10^{-17}$	$9.8x10^{-14}$

Note that the values of false synchronization in Table I assume random noise or data bits. If quasi-static data are being received there is a possibility that particular data words occurring at the frame rate could also cause false synchronization. Presently there are no automatic operations to decrease the probability for this type of false synchronization lock. The data user would have to recognize that the data were invalid and initiate a manual resynchronization.

The probability that a valid synchronization pattern is detected in data having a bit error probability q_b = BER and allowing up to E bits in error per pattern is shown in Eq(4).

$$p\ (k \leq E) = \sum_{k=0}^{E} \binom{26}{k} q_b^{\,k}\, p_b^{\,26-k}\ ,\quad p_b = 1-q_b \qquad (4)$$

The probability p $(k \leq E)$ takes on subscripts s_1, s_2, or c depending on the synchronization phase and mode, and E would have similar subscripts. Table II shows values of Eq(4) with BER's from $1x10^{-1}$ to $1x10^{-4}$ and allowed bit errors per pattern from 0 to 5.

TABLE II

Values of p for Eq(4) for various BER's and E bits in error per synchronization pattern.

E	BER 1×10^{-1}	BER 1×10^{-2}	BER 1×10^{-3}	BER 1×10^{-4}
0	0.0646 1082	0.7700 4314	0.9743 2241	0.9974 0324
1	0.2512 6429	0.9722 7669	0.9996 8015	0.9999 9675
2	0.5105 0523	0.9978 1123	0.9999 9744	0.9999 9999+
3	0.7409 4162	0.9997 7463	0.9999 9998	0.9999 9999+
4	0.8881 6487	0.9999 9447	0.9999 9999+	0.9999 9999+
5	0.9601 4068	0.9999 9980	0.9999 9999+	0.9999 9999+

Values from Tables I and II may be substituted in Eq(2) to determine the probability of successfully completing the search mode. For this computation, m is assumed to be 50; that is about 50 uncorrelated opportunities for a length of bits equal to the synchronization pattern are available in one telemetry data frame. Values of Eq(2) are shown in Table III.

TABLE III

The probability, P, and the average time, μ_1, for successful completion of the search mode.

BER	$E_{\emptyset_1}$ (bits)	$E_{\emptyset_2}$ (bits)	μ_1 (frames)	P MSFPT-1 System	P Dynatronics System
10^{-1}	5	4	1.04	0.884 803	0.963 674
10^{-1}	4	3	1.13	0.971 963	0.992 406
10^{-1}	3	2	1.35	0.993 963	0.997 034
10^{-1}	2	1	1.96	0.999 022	0.999 231
10^{-1}	1	C	3.98	0.999 841	0.999 915
10^{-1}	0	0	15.5	0.999 977	0.999 977
10^{-2}	1	0	1.03	0.999 996	0.999 997
10^{-2}	0	0	1.30	0.999 998	0.999 998

The number of allowable bit errors for the Dynatronics two-phase search mode are shown as $E_{\emptyset_1}$ and $E_{\emptyset_2}$ in Table III. Comparable errors for the MSFPT-1 single-phase search mode are indicated as $E_{\emptyset_1}$.

Many additional combinations of $E_{\emptyset_1}$ and $E_{\emptyset_2}$ allowed bit differences could have been used for Table III. The range of both preset values is from zero to nine bits. Additionally, there is no mechanical restriction limiting $E_{\emptyset_2}$ to be less than $E_{\emptyset_1}$.

The advantage of the two-phase search mode is most apparent when attempting to synchronize data with a BER of 1×10^{-1} or greater. For example, Table III shows P = 0.88 for a single-phase system allowing 5 bit errors in the detected pattern. The probability of successfully terminating the search mode with a two-phase system is 0.96. The two-phase search mode for this case has about an 8 percent greater chance to provide valid synchronization entry to the check mode. If the allowed bits in error, $E_{\emptyset_1}$, and $E_{\emptyset_2}$, are set equal there is no advantage in using the two-phase search mode.

The two phase search mode therefore improves the ability of the decommutator system to provide a valid synchronization as the number of allowed search mode error bits is increased. Increasing the allowed errors is necessary to decrease the average time to obtain a valid synchronization pattern detection. The average time, in telemetry frames, between valid patterns having a probability of detection shown in Table II is approximated by the reciprocal of P_{s_1}. Therefore, the average time, μ_1, for detecting a synchronization pattern allowing 2 bit errors is about 2 frames when the BER is 1×10^{-1}. If zero bit errors were allowed, about 16 frames would be expected to occur before a pattern meeting the allowable errors would be available for detection.

The synchronization criterion is to maximize the operational time in lock mode if data having an acceptable BER are available. Therefore, it is necessary to minimize the acquisition time for data having this BER. A run of valid synchronization patterns is necessary to complete the acquisition phase. One pattern in search and $F_c = r$ patterns in check must be satisfactorily detected.

The theory of recurrent events has been developed by Feller.[9] An application of this theory will be used to determine the average time, μ, in units of data frames required to complete the acquisition phase. Examples of this application and probability distributions for which the average values of Eq(5) apply are contained in Appendix II.

$$\mu_c(\text{frames}) = \frac{1-p_c^r}{q_c p_c^r} \tag{5}$$

In Eq(5), p_c is the probability of a successful event; i.e., the probability that a synchronization pattern contains E_c or less errors, $q_c = 1-p_c$, and r is the length of the run. A run of length r is defined as the number of consecutive valid pattern detections required in the check mode.

Values from Table II and values of r from 1 to 5 frames were used in Eq(5) to compute the average number of frames required to complete the check mode. These values are shown in Figure 5.

The minimum time required to complete a run of r valid patterns is r. Therefore, as the average time between r valid patterns approaches r there is less advantage in increasing p_c by allowing E_c to increase. Figure 5 shows that if $r = 5 = F_c$, about a minimum average time may be expected if $E_c = 0$ for a BER of 1×10^{-3}, $E_c = 2$ for a BER of 1×10^{-2}, and $E_c = 5$ for a BER of 1×10^{-1}. As r is reduced, the average time would more closely approach the minimum time.

The criterion to maximize lock time requires a value function to optimize the average time required to enter lock mode with valid synchronization as balanced against the time lost by entering the lock mode with invalid synchronization. Since the lock time is expected to be several minutes, several thousand telemetry frames, this value function would provide trivial answers for the range of bit errors per pattern and BER's considered. For example, if five frames were required in the acquisition phase allowing five errors per pattern, the probability of entering lock mode with invalid synchronization is about 1×10^{-13} (Table I) or one occurrence in about 5×10^{7} hours of receiving only noise bits.

Values from Figure 5 and Table I may be used to form Table IV showing average acquisition times and probabilities of false entry into the lock mode. Allowable bit errors per pattern are assumed equal in the search and check modes. This assumption is reasonable when considering that the BER is not expected to change radically during data reception except during multipath interference occurring at the beginning and ending of an orbital pass.

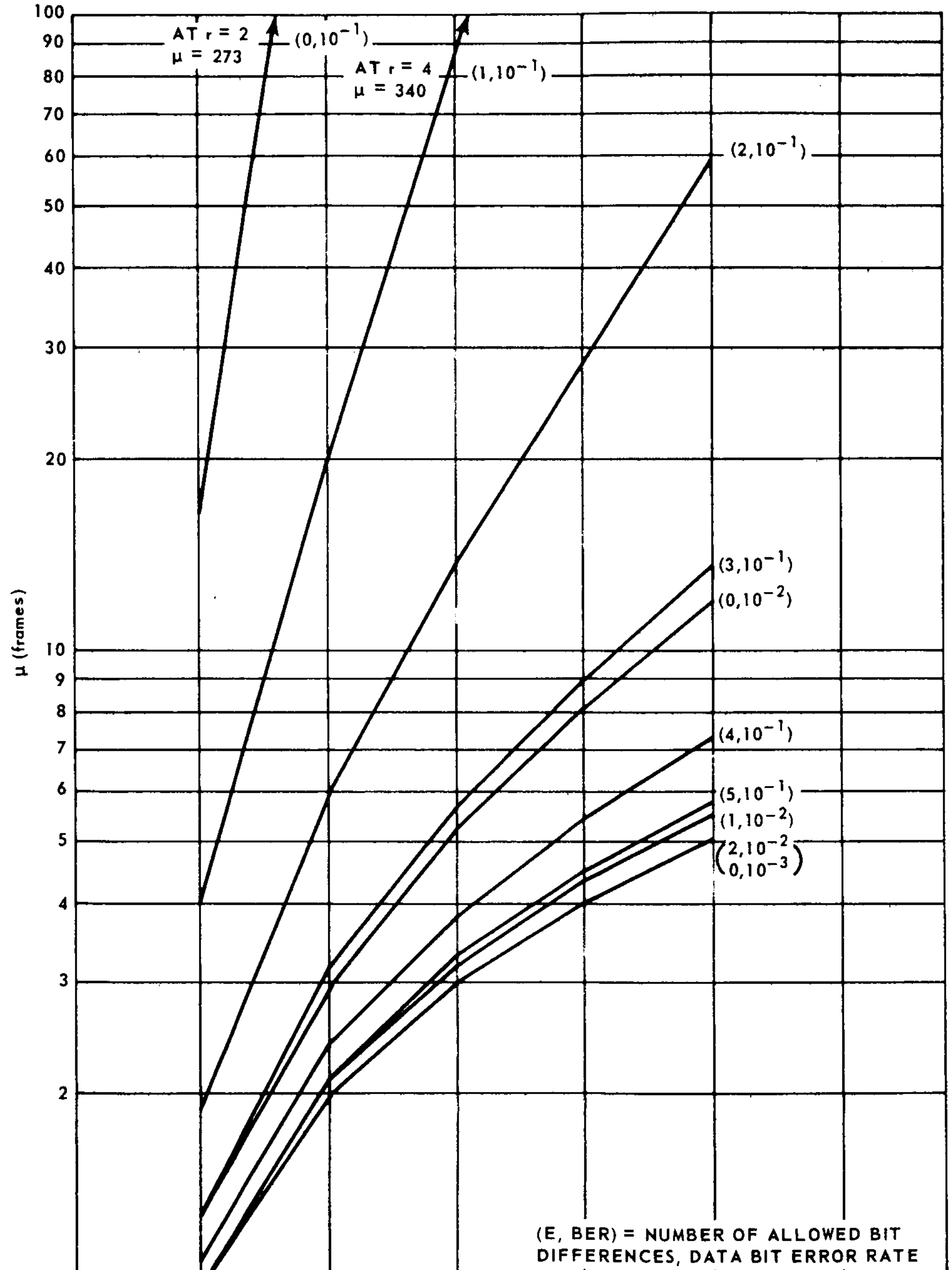


FIG. 5 AVERAGE TIME, μ, BETWEEN A RUN OF r GOOD FRAMES

TABLE IV

Average time to complete the acquisition phase

BER	$E_{\emptyset_1}$	E_c	F_c	$p_{f_c}^{r+1}$	μ(frames)	r+1
10^{-1}	5	5	4	$9.8x10^{-14}$	5.7	5
10^{-1}	4	4	3	$8.1x10^{-14}$	5.4	4
10^{-1}	3	3	2	$6.8x10^{-13}$	5.6	3
10^{-2}	2	2	2	$1.2x10^{-15}$	3.0	3
10^{-2}	1	1	1	$6.5x10^{-13}$	2.1	2
10^{-2}	0	0	1	$8.9x10^{-16}$	3.0	2
10^{-3}	0	0	1	$8.9x10^{-16}$	2.0	2

Table IV indicates a minimum average acquisition time with a BER = $1x10^{-1}$ would be obtained by allowing 4 bit errors per pattern in the search and check modes and setting F_c to 3. These settings provide a probability of $8x10^{-4}$ of entering lock mode with invalid frame synchronization. Similarly, allowing one bit error per pattern in the search and check modes and F_c =1 would minimize acquisition time for 1 x 10^{-2} BER data providing a false lock probability of $6.5x10^{-13}$. Data having a BER of $1x10^{-3}$ or less could be required to have only one check pattern with zero errors in the search and check modes to provide p_{f_c} = $8.9x10^{-16}$. Since the acquisition times are a trivial percent of the total expected lock time, the average acquisition times may be extended by requiring less allowable errors and more check frames. Such a procedure would cause a greater average acquisition time for undesirable data than for acceptable data.

The recommended acquisition settings for a maximum acceptable BER of $1x10^{-2}$ would be $E_{\emptyset_1}$ = 1, $E_{\emptyset_2}$ = 0, E_c = 1, and F_c = 3. These settings would provide for an average acquisition time of 4.3 frames and about 4 frames for data having a lower BER. The probability of false entry into lock mode operation would be $4.2x10^{-25}$. Additionally, data of a lower quality, BER = 1×10^{-1}, would require an average acquisition time of about 340 frames (about 7 seconds for Apollo data). Therefore, data with greater BER's than 1×10^{-2} would spend most of the acquisition time in the search to check mode loop shown in Figure 4.

If data having a BER of 1×10^{-3} were the acceptable value, the recommended acquisition settings would be $E_{\phi_1} = 0 = E_{\phi_2}$, $E_c = 0$, and $F_c = 4$. These settings would provide an average acquisition time of about 5 frames and a probability of entering lock mode with invalid synchronization of about 2.4×10^{-38}. Data having greater BER's would require increased acquisition times. Data with a BER of 1×10^{-2} would require an average of 13 frames, and several thousand frames would be required for data having a BER of 1×10^{-1}.

The actual acquisition time would be on the order of one half frame greater than the values stated. Since this value would be about the same for all acquisition times it was not added to the listed values. The one half frame additional time results from the assumption that on the average the acquisition phase starts in the middle of the telemetry frame after which a valid synchronization pattern would occur.

4. LOCK MODE SYNCHRONIZATION METHODS

Perhaps the most important decommutator settings are those providing termination of lock mode operation. Data are generally provided to the peripheral systems only when the synchronization subsystem has advanced to operation in this mode. Therefore, it is desirable that the greatest percentage of operational time be spent providing output data when a usable quality of data is available.

Decommutator settings for the search and check modes were not too critical if the E and F switches were set within a reasonable range. Table I and Figure 5 indicated probabilities of entering lock with false synchronization and average times required to enter with valid synchronization. Although optimized settings were given, variations from those settings would not materially impair satisfactory decommutator operation. Analysis of lock mode operation will also indicate that the E_L and F_L switches may provide satisfactory operation over a reasonable range of switch settings. Termination of lock mode operations requires that $F_L = \rho$ consecutive invalid frame synchronization patterns be recognized prior to one valid pattern. A valid pattern is one containing E_L or less bits difference from the true synchronization pattern.

The average time between runs of consecutive invalid synchronization patterns may be determined from Eq(6). It will be noted that Eq(6) is similar to Eq(5) in that the p's and q's have been interchanged. Equation(5) provided the average time between valid runs for the acquisition phase, and Eq(6) provides the average time between invalid synchronization runs for the lock mode. Examples of the probability distribution for which Eq(6) is valid are given in Appendix II.

$$\mu_L(\text{frames}) = \frac{1-q_L^{\rho}}{p_L q_L^{\rho}} \tag{6}$$

Values for p_L are provided in Table II, $q_L = 1-p_L$, and $\rho = F_L$ is the number of consecutive times an invalid pattern detection is required to terminate lock mode operation.

Figure 6 shows average times in seconds expected for various BER's and lock mode switch settings. The conversion between the units of telemetry frames provided by Eq(6) and seconds is determined by dividing the values of Eq(6) by 50 frames per second. Considering that an average maximum length orbital pass would be on the order of 10 minutes, it would be desirable to limit the average time between invalid pattern runs to values greater than this time. Figure 6 shows several switch settings for different BER's that have average times greater than 600 seconds. For example, if $E_L = 2$ and $F_L = 2$, an average time of about 4,100 seconds would be expected before lock termination with data having a BER of 10^{-2}.

However, the probability of maintaining lock operation for the average time is only about 70 percent.* In general, a higher probability of maintaining lock operation would be desirable assuming that valid data were available. Therefore, one would select a μ_L to be several times the expected data reception time to obtain this higher probability.

The probability of maintaining synchronization lock for some time t can be reasonably approximated by Eq(7)*. The restricting conditions for this approximation are (1) that $t \approx \mu_L/10$, and (2) that $\mu_L >> 1$.

$$t_n \approx 1 - \frac{t}{\mu_L} \tag{7}$$

The values for μ_L are from Figure 6 in units of seconds, and t is the time of required lock operation in units of seconds. In general μ_L will be on the order of several thousand frames and will satisfy the approximation condition (2).

Assuming that a 90 percent probability of maintaining synchronization lock for a given length of time t would be a desirable minimum, selection of any E_L and F_L switches having an expected μ_L at least 10 times t would meet the approximation condition (1). The previous example indicated $\mu_L = 4{,}100$. If F_L were set to 3 requiring only one frame increase in the invalid pattern run, $\mu_L = 1.9 \times 10^6$ seconds. Thus, $t_n \approx 1 - \frac{600}{1.9\text{x}10^6} =$ 0.9997 or it could be expected that the probability of

*Examples of the probability distribution are given in Appendix II. Equation (7) is also developed in Appendix II and shown on page AII-16.

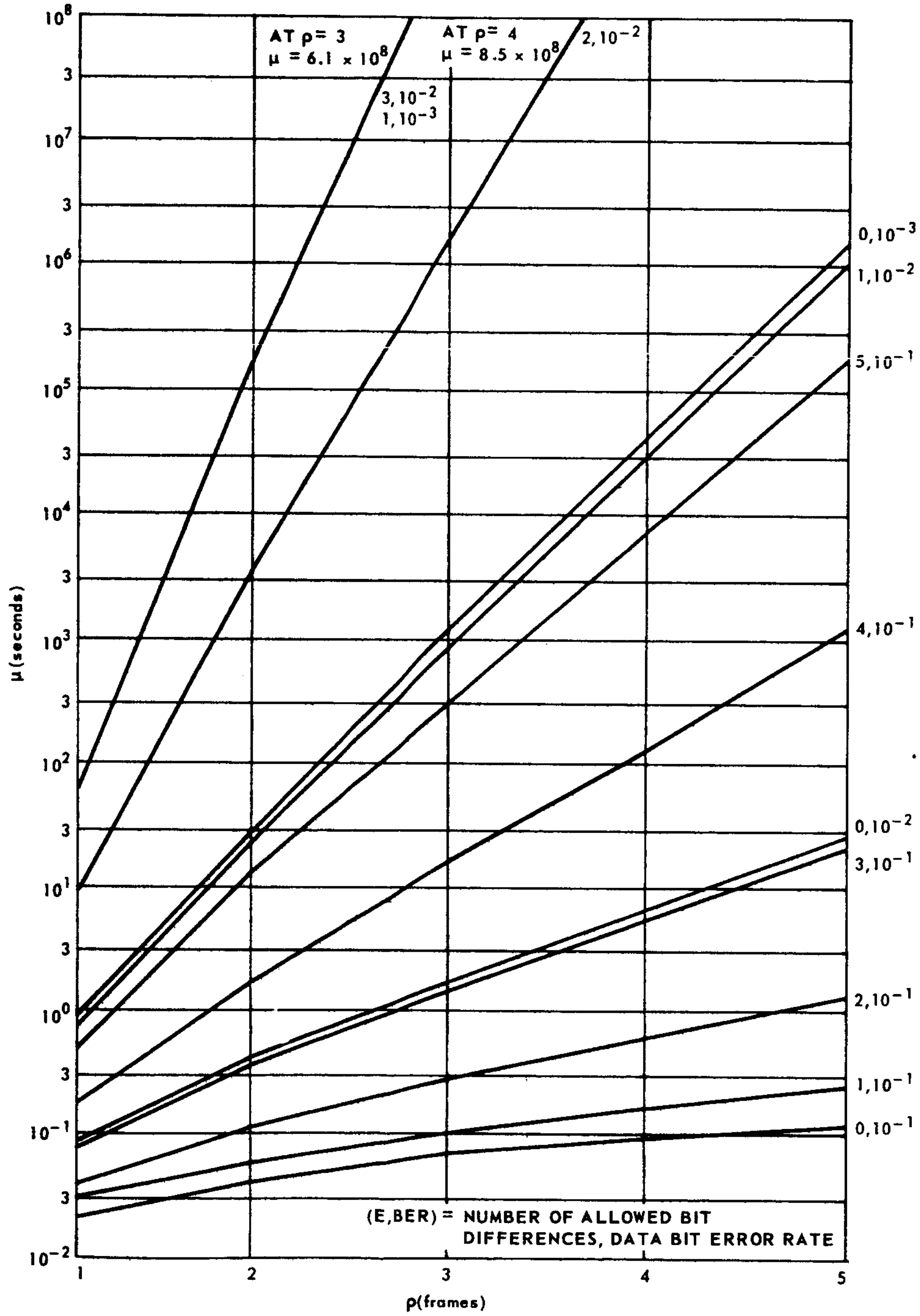


FIG. 6 AVERAGE TIME, μ, BETWEEN RUNS OF ρ BAD FRAMES

maintaining synchronization lock would be about 99.97 percent provided that the data had a maximum BER of 1×10^{-2} during a 600 second data run.

There are four basic reasons for leaving lock mode operation. (1) Loss of valid lock caused by the occurrence of an invalid pattern run. (2) Loss of valid lock because the data quality drops below an acceptable limit. (3) Loss of invalid lock caused by false synchronization entry to lock mode operation. (4) Loss of invalid lock because of a loss of bit synchronization.

The probability of loss of valid lock with a minimum acceptable data quality is predictable by t_n. This value can be increased at the expense of increasing μ_L for data of a lower quality. Also increasing t_n may require increasing F_L. A longer run of invalid patterns would therefore occur because of a loss in bit synchronization or an invalid entry to the lock mode because of false frame synchronization.

Loss of invalid lock because of false entry into lock mode or loss of valid bit synchronization occurring when in lock mode requires that a minimum time occur prior to returning to search operation. In general this time would be set by the F_L switch. Since F_L will be such a small percentage of an expected data reception time, and the probabilities of false lock entry and loss of bit synchronization should be quite small, an optimizing equation relating these functions should show a trivial effect on invalid lock operational time.

The synchronization criterion for this report requires that if the data quality falls below a desirable value, lock mode operation is terminated as soon as possible. The average time μ_L for data of a higher BER than acceptable will be used to define the lock mode E_L and F_L switch settings.

Table V shows values of t_n for data runs, t, of 600 seconds as a function of the E_L and F_L switch settings (E,F), and data having a BER of 1×10^{-2}. Also shown is the average lock time, $\mu_{10^{-1}}$, for data having a BER of 1×10^{-1}. The ratio of $t_n/\mu_{10^{-1}}$ provides a measure of the best choice of settings for a given value of t_n.

TABLE V

Lock mode switch settings providing a given value of t_n for a 600 second data run for data having a BER of 1×10^{-2}.

E_L, F_L	t_n (approximate)	$\mu_{10}-1$ (seconds)	$t_n/\mu_{10}-1$
5,2	0.9999 9999	13.06	0.077
4,2	0.9999 99	1.78	0.562
3,3	0.9999 99	1.52	0.658
2,3	0.9997	0.29	3.447
1,5	0.9995	0.26	3.844
3,2	0.998	0.38	2.626
5,1	0.994	0.50	1.988
1,4	0.982	0.17	5.776

The E_L and F_L switch settings providing the best balance between t_n and $\mu_{10}-1$ for an expected data run of 600 seconds would be $E_L = 2$, $F_L = 3$, or $E_L = 1$, $F_L = 5$. The settings where $F_L = 3$ would provide a slightly greater value of t_n but $\mu_{10}-1$ is increased about 0.03 second from the $F_L = 5$ settings. However, the lower number of frames required for the first switch settings would allow operation in the lock mode to be terminated two frames sooner if bit synchronization were lost or a false synchronization occurred allowing entry to lock mode operation.

In general, Table V indicates several switch settings that will provide satisfactory lock mode operation. If the expected time of a data run were increased to 6000 seconds (about 17 hours) the previously stated best settings would still provide better than a 99 percent probability of maintaining synchronization lock with data having a BER of 1×10^{-2} [from Eq(7)].

Table VI shows the values similar to those of Table V but for a maximum acceptable BER of 1 x 10^{-3}. The best switch settings would be $E_L = 1$, $F_L = 2$ providing a reasonable probability of maintaining lock operation while minimizing the lock time for data having a BER of 1×10^{-2}. If a data run time of 6000 seconds was expected, settings of $E_L = 3$, $F_L = 1$ would provide better than 99 percent probability of maintaining lock keeping $\mu_{10^{-2}}$ to 63 seconds. Settings of $E_L = 0$, $F_L = 5$ would provide about the same probability of maintaining lock operation but would allow $\mu_{10^{-2}}$ to increase about 64 times to over 4,000 seconds.

TABLE VI

Lock mode switch settings providing a given value of t_n for a 600 second data run for data having a BER of 1×10^{-3}

E_L, F_L	t_n (approximate)	$\mu_{10^{-2}}$ (seconds)	$t_n/\mu_{10^{-2}}$
2,2	0.9999 998	4,142	0.0002
1,3	0.9999 99	968	0.0010
0,5	0.9997	4,032	0.0002
3,1	0.9994	63	0.0159
1,2	0.998	27	0.0369
0,4	0.987	93	0.0106
2,1	0.92	9	0.1022

5. CONCLUSIONS

A description of the operation of two PCM decommutator frame synchronization subsystems has been presented. The decommutators are part of a real time information system supplying data to the NASA Manned Space Flight Tracking Network.

Received telemetry data have frame synchronization patterns which must be validly detected in order to have correct decommutation operation. Error free detection of the patterns is precluded since perfect data transmission is not available. Thus, a frame synchronization threshold is established by allowing some number of bit differences between a detected pattern and the true pattern. Additionally, the number of required sequential valid patterns in check mode and invalid patterns in lock mode influence the synchronization threshold.

A synchronization criterion was selected that is believed best for decommutator usage in the present information system. This criterion is to minimize the average time required for synchronization acquisition if data having a minimum quality set by a maximum acceptable bit error rate (BER) are available. If data of a poorer quality are present it is then desired to increase the acquisition time to keep the subsystem from lock operation. The criterion also seeks to have a high probability of maintaining synchronization if the data quality is acceptable. If data of a poorer quality occur during synchronization maintenance it is desired that the synchronization subsystem revert to an inferior mode of operation in a minimum time.

Equations relating probabilities and average times of obtaining and maintaining frame synchronization were presented. Figures showing average synchronization acquisition and maintenance times provide insight into the effects of varying the threshold settings.

Additionally, the criterion was used to select the "allowed pattern bit errors" and "number of patterns" switch settings under the assumption of maximum acceptable BER's of 1×10^{-2} and 1×10^{-3} for a data reception time of about 10 minutes. A high probability of maintaining synchronization was also indicated for a data reception time of 100 minutes with the switch settings for 1×10^{-2} BER data. A change in switch settings was necessary for the longer data time when the maximum acceptable BER was 1×10^{-3}.

The techniques used in this report may be extended by digital computer computation. The approximation used to determine the probability of maintaining synchronization could be replaced by the actual probability from distributions presented in Appendix II. Several criteria could be programmed and the average times or probabilities presented with decommutator switch settings. Such a general program would seem to be a valuable tool for anyone using PCM decommutators.

All analysis in this report presumed perfect bit synchronization. In practice this would not be valid for poor signal to noise ratios and particular bit synchronizers. The probability of bit synchronization should be considered in any future effort to establish frame synchronization thresholds.

GLOSSARY OF TERMS

This glossary is provided as a reference for the mathematical terminology and symbols used in this paper.

A) Probabilistic Events (Search Mode)

S = The true sync word exists at a random independent trial.

S' = The true sync word does not exist at a random independent trial.

T = The true sync word is detected at a random independent trial.

F = A false sync word is detected at a random independent trial.

ξ = A sync word (true or false) is detected at a random independent trial.

N = No sync word (true or false) is detected at a random independent trial.

$P(T/S)$, $P(F/S')$etc. = The conditional probability of the event.

B) Symbols (Search Mode)

m = The number of independent trials in the sync word search process per data frame.

M = The number of bits per data frame ($M = 1024$)

n = The number of bits in the sync word ($n = 26$)

E_{ϕ_1} = The allowed number of bits in error in the sync word correlation process for phase 1 of the search mode. E_{ϕ_1} is a switch selected variable.

E_{ϕ_2} = The allowed number of bits in error in the sync word correlation process for phase 2 of the search mode. E_{ϕ_2} is a switch selected variable.

q_b = The bit error probability, ($q_b = 10^{-1}, 10^{-2}, 10^{-3}$ etc.). The term Bit Error Rate, BER, is used synonomously with q_b in the text even though the bit error rate is time dependent on the bit rate whereas q_b is not.

$p_b = 1 - q_b$

p_{s_1} = The probability that the true sync word contains $E_{\emptyset_1}$ or less errors ($p_{s_1} = P(T/S)$)

$$p_{s_1} = p(T/S) = \sum_{k=0}^{E_{\emptyset_1}} \binom{26}{k} q_b^{\,k}\, p_b^{\,26-k}, \text{ for } n = 26$$

$q_{s_1} = 1 - p_{s_1}$

p_{f_1} = The probability that a random group of n bits appears as a sync word with $E_{\emptyset_1}$ or less errors.

($p_{f_1} = P(F/S') = 2\sum_{k=0}^{E_{\emptyset_1}} \binom{26}{k} (\frac{1}{2})^{26}$, for n = 26), allowing for complement agreements

$q_{f_1} = 1 - p_{f_1}$

μ_1 = The average number of trials resulting in no sync before the first trial resulting in sync (true or false).

μ_{t_1} = The average number of trials resulting in no sync or false sync before the first trial resulting in true sync.

μ_{f_1} = The average number of trials resulting in no sync or true sync before the first trial resulting in false sync

p_{s_2} = The probability that the true sync word contain $E_{\emptyset_2}$ or less errors

$$p_{s_2} = \sum_{k=0}^{E_{\emptyset_2}} \binom{26}{k} q_b^{\,k}\, p_b^{\,26-k}, \text{ for } n = 26)$$

$q_{s_2} = 1 - p_{s_2}$

p_{f_2} = The probability that a random group of n bits appears as a sync word with E_{ϕ_2} or less errors

$$p_{f_2} = 2 \sum_{k=0}^{E_{\phi_2}} \binom{26}{k} \left(\tfrac{1}{2}\right)^{26}, \text{ for } n = 26)$$

$q_{f_2} = 1 - p_{f_2}$

ψ_k = A sequence of m-1 trials S' S'...S...S', with the true sync word existing at the kth trial.

P = The probability that the true sync word is found in the search mode

Q = The probability that a false sync word is found in the search mode.

C) Symbols (Check and Lock Modes)

$\mathcal{E}$ = An event satisfying certain properties. For example $\mathcal{E}$ is a run of 3 consecutive failures in a sequence of trials. (3 consecutive bad sync words in 3 consecutive frames)

u_n = The probability that the event occurs at the nth trial

$U(s) = u_o + u_1S + u_2s^2 + \ldots$ = the generating function of the probabilities u_n

g_n = The probability that an event $\mathcal{E}$ occurs for the first time at the nth trial. When the symbol g is used the event will be a success run (e.g., consecutive frames containing a "good" sync word) of a specified length.

$G(s) = g_o + g_1s + g_2s^2 + \ldots$ = The generating function of the probabilities g_n. In rational fraction form, $G(s) = \frac{P(s)}{Q(s)}$

b_n = The probability that an event $\mathcal{E}$ occurs for the first time at the nth trial. When the symbol b is used the event will be a failure run (e.g. consecutive frames containing a "bad" sync word) of a specified length.

$B(s) = b_0+b_1s+b_2s^2+\ldots$ = the generating function of the probabilities b_n. In rational fraction form, $B(s) = \frac{X(s)}{Y(s)}$

E_c = The allowed number of bits in error in the sync word correlation process for the check mode. E_c is a switch selected variable.

E_L = The allowed number of bits in error in the sync word correlation process for the lock mode. E_L is a switch selected variable.

p_c = The probability that the sync word contains E_c or less errors in the check mode.

$$p_c = \sum_{k=0}^{E_c} \binom{26}{k} q_b^k \ p_b^{26-k}$$

$q_c = 1-p_c$

p_L = The probability that the sync word contains E_L or less errors in the lock mode.

$$p_L = \sum_{k=0}^{E_L} \binom{26}{k} q_b^{\ k} \ p_b^{26-k}$$

$q_L = 1-p_L$

p_{f_c} = The probability that a random group of 26 bits appears as a sync word with E_c or less errors.

p_{f_L} = The probability that a random group of 26 bits appears as a sync word with E_L or less errors.

$\nu_n = t_n$ = The probability that an event $\mathcal{E}$ does not occur in n trials (i.e. the probability that the event $\mathcal{E}$ occurs after the nth trial). The t_n are the "tail" probabilities for the b_n and give the probability that a failure run of specified length does not occur until after the nth trial.

t_n = The probability that an event does not occur in n trials where the event is a failure run of specified length with each trial having probability p_{f_L} (invalid sync).

$T(s) = t_o+t_1s+t_2s^2+\ldots$ = The generating function of the probabilities t_n.

h_n = The probability that an event does not occur in n trials. The h_n are the "tail" probabilities for g_n and give the probability that a success run of specified length does not occur until after the nth trial.

$H(s) = h_o+h_1s+h_2s^2 +\ldots$ = The generating function of the probabilities h_n

$F_c = r$ = The number of consecutive successes, i.e. "good" sync words, required in the check mode. (length of a success run) $r = 1,2,3,\ldots$etc. $F_c = r$ is a switch selected variable.

μ_g = The average trial at which the first success run of length r is completed.

σ_g^2 = The variance of the trial number at which the first success run of length r is completed.

$F_L = \rho$ = The number of consecutive failures, i.e. "bad" sync words, allowed in the lock mode. (length of a failure run), $\rho = 1,2,3,\ldots$etc. $F_L = \rho$ is a switch selected variable.

μ_b = The average trial at which the first failure run of length ρ is completed.

σ_b^2 = The variance of the trial number at which the first failure run of length ρ is completed.

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APPENDIX I

Search Mode Analysis

The basic objective of this appendix is to provide insight into the effects of changing the switch controlled variables on the probabilities of true and false synchronization and the average time spent in the search mode. In order to describe search mode operation in a simplified way the following approximating analysis will be used. First several basic events are defined:

S = the true sync word exists at a random independent trial

S' = the true sync word does not exist at a random independent trial

T = the true sync word is detected at a random independent trial

F = a false sync word is detected at a random independent trial

ξ = a sync word (true or false) is detected at a random independent trial

N = no sync word (true or false) is detected at a random independent trial

m = the number of independent trials per data frame for locating the true sync word. The event S, existence of the true sync word, occurs at one of the m trials.

The use of m independent trials per frame allows simplified analysis for the purpose of illustration and approximation. For a frame length of M bits and a sync word length of n bits, $n < M$, $\frac{M}{n} \leq m \leq M - (n-1)$. The upper bound on m corresponds to having dependence in the sync word search correlation process at the complete overlap position only. A shift of one bit causes the next trial to be independent. The lower bound corresponds to having independence only after the kth group of n bits has been shifted out of the correlator and the k + 1th group is entered. From the standpoint of minimizing the number of possibilities per data frame for false sync., a sync word choice tending toward the lower bound would be desirable. The subject of sync word format is discussed in several references[2,3,4] and it is not our purpose to delve into the matter. When a numerical value is needed for m it will be assumed that about a 20% overlap in the correlator allows an "independent" trial. For M = 1024 and n = 26, the values which apply to this study, this gives $n \approx 50$. It is thereby tacitly assumed that an "adequate" choice of sync word format has been made. The following probabilities will be assigned to the events previously defined.

$P(S) = \frac{1}{m}$; $P(T/S) = p_{s_1}$ = the probability that the true sync word contains at most E errors.

and $P(T/S') = 0$.

For search phase 1, $p_{s_1} = \sum_{k=0}^{E_{\phi_1}} \binom{26}{k} q_b^k \, p_b^{26-k}$, $q_{s_1} = 1 - p_{s_1}$

q_b = bit error probability

$q_b = 10^{-1}, 10^{-2}$ etc., $p_b = 1 - q_b$

$P(S') = \frac{m-1}{m}$; $P(F/S') = p_{f_1}$ = the probability that random bits occur as a sync word with at most E_{ϕ_1} errors.

$q_{f_1} = 1 - p_{f_1}$

and $P(F/S) = 0$

for search phase 1

$$p_{f_1} = 2 \sum_{k=0}^{E_{\phi_1}} \binom{26}{k} \left(\tfrac{1}{2}\right)^{26},$$ allowing for complement agreements, $q_{f_1} = 1 - p_{f_1}$

$P(T,S) = \frac{1}{m} p_{s_1}$

and since $P(S/T) = 1$

$$P(T) = \frac{1}{m} p_{s_1}$$

$P(F,S') = \frac{m-1}{m} \; p_{f_1}$

and since $P(S'/F) = 1$

$P(F) = \frac{m-1}{m} p_{f_1}$

$P(\xi) = \frac{1}{m} \; p_{s_1} + \frac{m-1}{m} p_{f_1}$

$P(N) = P(N/S)\,P(S) + P(N/S')\,P(S') = \frac{1}{m} \; q_{s_1} + \frac{m-1}{m} \; q_{f_1}$

and at any trial: $P(\xi) + P(N) = 1$

using $P(\xi/T)\,P(T) = P(T/\xi)\,P(\xi)$

$$P(T/\xi) = \frac{\frac{p_{s_1}}{m}}{\frac{p_{s_1}}{m} + \frac{(m-1)}{m} p_{f_1}} = \frac{p_{s_1}}{p_{s_1} + (m-1) p_{f_1}} = \frac{1}{1 + (m-1)\frac{p_{f_1}}{p_{s_1}}} \quad (1)$$

$P(T/\xi)$ = the probability that given a sync word is detected, it is the true sync word. (i.e. the probability that phase 1 of the search mode terminates with the true sync. word found).

using $P(\xi/F)\ P(F) = P(F/\xi)\ P(\xi)$

$$P(F/\xi) = \frac{\frac{(m-1)}{m} p_{f_1}}{\frac{p_{s_1}}{m} + \frac{(m-1)}{m} p_{f_1}} = \frac{(m-1)\ p_{f_1}}{p_{s_1} + (m-1) p_{f_1}} = \frac{1}{1 + \frac{p_{s_1}}{(m-1) p_{f_1}}} \quad (2)$$

$P(F/\xi)$ = the probability that given a sync word is detected, it is a false sync word. (i.e. the probability that phase 1 of the search mode terminates with a false sync word found)

Using $P(N) + P(\xi) = 1 = \frac{1}{m}\ q_{s_1} + \frac{m-1}{m}\ q_{f_1} + \frac{1}{m}\ p_{s_1} + \frac{m-1}{m}\ p_{f_1}$

let $P(N) = q'$
$P(\xi) = p'$

then via a geometric distribution approach:

$$p' + q'p' + q'^2p' + q'^3p' + .. = p' \left(\frac{1}{1-q'}\right) = 1, \text{ this implies}$$

that sooner or later a sync word will be detected and phase 1 will terminate. For the values of p_{s_1} and p_{f_1} pertinent to this study it is highly probable that phase 1 will terminate in a "reasonable" number of trials.

Again via the geometric distribution, μ_1 = the average number of "no sync" trials before the first "sync" trial

$$\mu_1 = q'/p' = \frac{q_{s_1} + (m-1)\ q_{f_1}}{p_{s_1} + (m-1)\ p_{f_1}} \quad (3)$$

To check μ_1 for $q_{s_1} = 0$, $q_{f_1} = 1$, the ideal case, is misleading since $\mu_{1_{ideal}} = m-1$, implying one frame interval on the average before the true sync word is found. It is more intuitive to expect $\mu_{1_{ideal}} = \frac{m-1}{2}$, implying that a half frame will pass on the average before the first sync word detection.

If $q_{s_1} = 0$ and $q_{f_1} = 1$, however, the probabilities $P(S) = \frac{1}{m}$, and $P(S') = \frac{m-1}{m}$

do not remain fixed but change with each trial:

$P(S_1) = \frac{1}{m}$ $P(S'_1) = \frac{m-1}{m}$

$P(S_2) = \frac{2}{m}$, $P(S'_2) = \frac{m-2}{2}$

$P(S_3) = \frac{3}{m}$, $P(S'_3) = \frac{m-3}{m}$

⋮

etc.

The trials are now dependent since for each "no sync" trial it is certain that the true sync word did not exist at that trial and thereby becomes more probable at the next trial. Adding the $P(S_k)$, $\sum_{k=1}^{m} \frac{k}{m}$, gives $\frac{m-1}{2}$ as expected for the ideal case. A similar "averaging" process would allow the factor m-1 to be replaced by $\frac{m-1}{2}$ in equations 1) and 2) and in the equations which follow for "near ideal" cases. However, the ratios, $\frac{p_{f_1}}{p_{s_1}}$, $\frac{p_{s_2}}{p_{f_2}}$, etc, have a stronger bearing on how quantities will vary and the term $(m-1)p_f$ implies the number of opportunities for false sync in a frame. For these reasons the notion of independent trials will be adhered to with the resulting factor, m-1.

Also by similar reasoning to that used in obtaining equation 3:

μ_{t_1} = the average number of "no sync" or "false sync" trials before the first "true sync". trial.

$$\mu_{t_1} = \frac{q_{s_1} + (m-1)}{p_{s_1}} \approx \mu_1, \text{ for } p_{f_1} \to 0$$

and

μ_{f_1} = the average number of "no sync" or "true sync" trials before the first "false sync" trial.

$$\mu_{f_1} = \frac{q_{f_1}}{p_{f_1}} + \frac{1}{(m-1)p_{f_1}}$$

for $p_{f_1} \to 0$

$$\frac{\mu_{f_1}}{\mu_{t_1}} \approx \frac{p_{s_1}}{(m-1)p_{f_1}}$$

equation 4) $$\frac{P(T/S)}{P(F/S)} = \frac{p_{s_1}}{(m-1)\, p_{f_1}}$$

Eqtn 4) implies that for "good" values of the probabilities ($p_{f_1} \to 0, p_{s_1} \to 1$) it is much more probable, relatively, that phase 1 terminate with the finding of the true sync word.

The outcomes of phase 2 of the search mode will now be considered as conditional upon whether phase 1 terminated with the finding of the true sync word or with the finding of a false sync word. Following the termination of phase 1, the sync word search process will continue for up to m-1 trials with a different bit disagreements allowed criterion per trial from that of phase 1. For phase 2:

$$p_{s_2} = \sum_{k=0}^{E_{\phi_2}} \binom{26}{k} q_b^{\,k}\, p_b^{\,26-k}; \quad q_{s_2} = 1-p_{s_2} \quad \text{and} \quad p_{f_2} = 2\sum_{k=0}^{E_{\phi_2}} \binom{26}{k} \left(\tfrac{1}{2}\right)^{26}; \quad q_{f_2} = 1-p_{f_2}$$

1) Given that phase 1 terminated with the finding of the true sync word the probability that phase 2 does not terminate with the finding of a false sync word before m-1 trials are completed is $q_{f_2}^{\,m-1}$.

$q_{f_2}^{\,m-1}$ = the probability that all m-1 trials result in "no sync" (for all the m-1 trials only a false sync, S', can exist since it is given that phase 1 terminated with the finding of the true sync word).

and $1-q_{f_2}^{\,m-1}$ = the probability that at least one false sync word is found and thereby phase 2 terminates with false sync. (At most m-1 trials are required).

2) Given that phase 1 terminated with the finding of a false sync word, sequences of the following form must be considered:

$$\begin{aligned}
\psi_1 &= S\ S'\ S'\ S'\ S' \ldots S' \\
\psi_2 &= S'\ S\ S'\ S'\ S' \ldots S' \\
\psi_3 &= S'\ S'\ S\ S' \ldots\ldots S' \\
&\ \ \vdots \\
\psi_{m-1} &= S'\ S'\ S'\ S' \ldots\ldots S
\end{aligned}$$

There are m-1 such sequences since S can occur at any one of m-1 places. Each sequence represents m-1 trials and the probability of each sequence is $\frac{1}{m-1}$.

The probabilities for the outcomes of phase 2 conditional upon phase 1 terminating with a false sync word will now be shown for the m-1 sequences. The three outcomes are: 1) the true sync word is found, 2) no sync word is found, 3) another false sync word is found. Outcome 1) leads to true sync whereas outcomes 2) and 3) lead to false sync.

i) for the sequence $\psi_1 = S\ S'\ S'\ S' \ldots S'$, $P(\psi_1) = \frac{1}{m-1}$

1) p_{s_2}, for true sync word found

2) $q_{s_2} q_{f_2}^{m-2}$, for no sync word found

3) $q_{s_2} p_{f_2} + q_{s_2} q_{f_2} p_{f_2} + q_{s_2} q_{f_2}^{2} p_{f_2} + \ldots q_{s_2} q_{f_2}^{m-3} p_{f_2}$

$= q_{s_2} p_{f_2} (1 + q_{f_2} + q_{f_2}^{2} + \ldots q_{f_2}^{m-3})$

$= q_{s_2} (1 - q_{f_2}^{m-2})$, for a false sync word found

(note that $p_{s_2} + q_{s_2} q_{f_2}^{m-2} + q_{s_2} - q_{s_2} q_{f_2}^{m-2} = 1$, these are all the possible outcomes for ψ_1)

ii) for the sequence $\psi_2 = S'\ S\ S'\ S' \ldots S'$, $P(\psi_2) = \frac{1}{m-1}$

1) $q_{f_2} p_{s_2}$, for true sync word found

2) $q_{s_2} q_{f_2}^{m-2}$, for no sync word found

3) $p_{f_2} + q_{f_2} q_{s_2} p_{f_2} + q_{s_2} q_{f_2}^{2} p_{f_2} + q_{s_2} q_{f_2}^{3} p_{f_2} + \ldots q_{s_2} q_{f_2}^{m-3} p_{f_2}$

$= p_{f_2} + q_{s_2} q_{f_2} (1 - q_{f_2}^{m-3}) = 1 - q_{f_2} + q_{s_2} (q_{f_2} - q_{f_2}^{m-2})$, for a false sync word found.

(Again, $q_{f_2} p_{s_2} + q_{s_2} q_{f_2}^{m-2} + 1 - q_{f_2} + q_{s_2} q_{f_2} - q_{s_2} q_{f_2}^{m-2} = 1$, these are all the possible outcomes for ψ_2).

iii) for the sequence ψ_3 = S' S' S S' S' S'...S', $P(\psi_3) = \frac{1}{m-1}$

1) $q_{f_2}^{\ 2}\ p_{s_2}$, for true sync word found

2) $q_{s_2} q_{f_2}^{\ m-2}$, for no sync word found

3) $p_{f_2} + q_{f_2} p_{f_2} + q_{s_2} q_{f_2}^{\ 2} p_{f_2} + q_{s_2} q_{f_2}^{\ 3} + \ldots q_{s_2} q_{f_2}^{\ m-3}\ p_{f_2}$

$= p_{f_2}(1 + q_{f_2}) + q_{s_2} q_{f_2}^{\ 2} p_{f_2}(1 + q_{f_2} + q_{f_2}^{\ 2} + \ldots\ q_{f_2}^{\ m-5})$

$= (1 - q_{f_2}^{\ 2}) + q_{s_2}(q_{f_2}^{\ 2} - q_{f_2}^{\ m-2})$, for a false sync word found

etc.

⋮

m-1) for the m-1th sequence, ψ_{m-1} = S' S' S'...S, $P(\psi_{m-1}) = \frac{1}{m-1}$

1) $q_{f_2}^{\ m-2}\ p_{s_2}$, for true sync word found

2) $q_{s_2} q_{f_2}^{\ m-2}$, for no sync word found

3) $p_{f_2} + q_{f_2} p_{f_2} + q_{f_2}^{\ 2}\ p_{f_2} + \ldots q_{f_2}^{\ m-3}\ p_{f_2} + q_{s_2}(q_{f_2}^{\ m-2} - q_{f_2}^{\ m-2})$

$= (1 - q_{f_2}^{\ m-2}) + q_{s_2}(q_{f_2}^{\ m-2} - q_{f_2}^{\ m-2})$, for a false sync word found.

Now adding the probabilities gives:

$$\frac{1}{m-1}\Big\{ p_{s_2} + q_{f_2}\ p_{s_2} + q_{f_2}^{\ 3}\ p_{s_2} + \ldots q_{f_2}^{\ m-2}\ p_{s_2} + (m-1)\ q_{s_2} q_{f_2}^{\ m-2} + m-2$$
$$- q_{f_2}(1 + q_{f_2} + q_{f_2}^{\ 2} + \ldots q_{f_2}^{\ m-3})$$
$$+ q_{s_2}\ [1 + q_{f_2} + q_{f_2}^{\ 2} + \ldots\ q_{f_2}^{\ m-2} - (m-1)\ q_{f_2}^{\ m-2}]\Big\}$$
$$= \frac{p_{s_2}}{(m-1)p_{f_2}}\ (1 - q_{f_2}^{\ m-1}) + q_{s_2} q_{f_2}^{\ m-2} + \frac{m-2}{m-1} - \frac{q_{f_2}}{(m-1)p_{f_2}}\ (1 - q_{f_2}^{\ m-2})$$
$$+ \frac{q_{s_2}(1 - q_{f_2}^{\ m-1})}{(m-1)p_{f_2}}\ - q_{s_2} q_{f_2}^{\ m-2}$$

The probabilities for the outcomes of the m-1 trials after the termination of phase 1, given that phase 1 terminated with a false sync word are therefore:

1) $\frac{p_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}^{m-1})$, for finding the true sync word

2) $q_{s_2} q_{f_2}^{m-2}$, for finding no sync word

3) $\frac{m-2}{m-1} + \frac{q_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}^{m-1}) - \frac{q_{f_2}}{(m-1)p_{f_2}} (1-q_{f_2}^{m-2}) - q_{s_2} q_{f_2}^{m-2}$,

for finding a false sync word

Adding the three as a check:

$$\frac{p_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}^{m-1}) + q_{s_2} q_{f_2}^{m-2} + \frac{m-2}{m-1} + \frac{q_{s_2} (1-q_{f_2}^{m-1})}{(m-1)p_{f_2}} - \frac{q_{f_2} (1-q_{f_2}^{m-2})}{(m-1)p_{f_2}} - q_{s_2} q_{f_2}^{m-2}$$

$$= \frac{(1-q_{f_2}^{m-1})}{(m-1)p_{f_2}} - \frac{q_{f_2}}{(m-1)p_{f_2}} (1-q_{f_2}^{m-2}) + \frac{m-2}{m-1}$$

$$= \frac{1-q_{f_2}^{m-1} - q_{f_2} + q_{f_2}^{m-1}}{(m-1)\, p_{f_2}} + \frac{m-2}{m-1} = \frac{1}{m-1} + \frac{m-2}{m-1} = 1$$

This implies that all the outcomes for the up to m-1 trials of phase 2 have been accounted for, given that phase 1 terminated with a false sync.

Now combining the probabilities for the outcomes of phase 2, conditional upon the outcomes of phase 1 gives the overall probabilities of true and false sync as follows:

P = probability of true sync

equation 5)
$$P = \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} \cdot q_{f_2}^{m-1} + \frac{(m-1)p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \cdot \frac{p_{s_2}(1-q_{f_2}^{m-1})}{(m-1)\, p_{f_2}}$$

Q = probability of false sync.

equation 6).
$$Q = \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} \cdot (1-q_{f_2}{}^{m-1}) + \frac{(m-1)p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \cdot q_{s_2} q_{f_2}{}^{m-1}$$

$$+ \frac{(m-1)p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \cdot \left[\frac{m-2}{m-1} + \frac{q_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-1}) - \frac{q_{f_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-2}) - q_{s_2} q_{f_2}{}^{m-2} \right]$$

Equations 5) and 6) are in the form which shows the individual conditional probabilities. Adding equations 5) and 6) as a check shows their sum to equal one.

$$P + Q = \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} \left[q_{f_2}{}^{m-1} + (1-q_{f_2}{}^{m-1}) \right] + \frac{(m-1)p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \left[\frac{p_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-1}) \right.$$

$$+ \frac{m-2}{m-1} + \frac{q_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-1}) - \frac{q_{f_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-2})$$

$$\left. - q_{s_2} q_{f_2}{}^{m-2} + q_{s_2} q_{f_2}{}^{m-2} \right]$$

$$= \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} + \frac{(m-1)p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \left[\frac{m-2}{m-1} + \frac{p_{s_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-1}) + \frac{q_{s_2}(1-q_{f_2}{}^{m-1})}{(m-1)p_{f_2}} \right.$$

$$\left. - \frac{q_{f_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-2}) \right]$$

$$= \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} + \frac{(m-1)p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \left[\frac{m-2}{m-1} + \frac{(1-q_{f_2}{}^{m-1})}{(m-1)p_{f_2}} - \frac{q_{f_2}}{(m-1)p_{f_2}} (1-q_{f_2}{}^{m-2}) \right]$$

$$= \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} + \frac{(m-1)\ p_{f_1}}{p_{s_1} + (m-1)p_{f_1}} \left[\frac{m-2}{m-1} + \frac{1-q_{f_2}}{(m-1)p_{f_2}} \right]$$

$$= \frac{p_{s_1}}{p_{s_1} + (m-1)p_{f_1}} + \frac{(m-1)p_{s_1}}{p_{s_1} + (m-1)p_{f_2}} = 1$$

The fact that P + Q = 1 is still dependent upon the original condition that the event ξ, a sync word (true or false) is detected, occurs in phase 1. It was indicated earlier that the occurrence of ξ in a "reasonable" number of phase 1 trials is highly probable for the values of p_{s_1} and p_{f_1} of interest.

Equation 5) can be simplified algebraically to give:

equation 7) $$P = \frac{p_{s_1}}{p_{s_1} + (m-1)\, p_{f_1}} \left[q_{f_2}^{\,m-1} + \frac{p_{f_1}}{p_{s_1}} \cdot \frac{p_{s_2}}{p_{f_2}} \left(1 - q_{f_2}^{\,m-1}\right) \right]$$

and rather than simplifying equation 6), the fact that Q = 1-P can be utilized more conveniently.

Equation 7) allows examination of the effect of setting the two different error criteria (phase 1 and phase 2) on the probability of true sync. The ratios $\frac{p_{f_1}}{p_{s_1}}$ and $\frac{p_{s_2}}{p_{f_2}}$ are the interesting variables since they are a function of the bit error rate and the number of errors allowed in the search mode correlation process, the two error criteria being different in general for phase 1 and 2.

APPENDIX II

Check and Lock Mode Analysis

The following material outlines and gives examples of the mathematical techniques used in obtaining numerical values pertaining to the discussions of average time in the check and lock modes as given in sections 3 & 4. The approach follows the methods used to analyze "recurrent events" given in the previously cited reference, An Introduction to Probability Theory and Its Applications, by William Feller, Volume I second edition John Wiley and Sons. For complete derivations, proofs, examples etc. that reference is recommended. The basic objective of this appendix is to provide insight into the effects of changing the switch controlled variables on the probabilities of valid check and on the average time spent in acquisition and lock. The idea of a recurrent event is used to describe a particular pattern of successive outcomes repeating itself at varying intervals during independent Bernoulli trials. The patterns of particular interest in this report are "success runs of length r" and "failure runs of length ρ". Each of these recurrent events has the property of consecutive identical outcomes of "failure" or "success" for a specified number of trials forming a "run". The examination of each frame sync word for bit errors constitutes a trial. Failure or success for that frame depends upon the number of errors in each sync word exceeding or being equal to or less than a preset number of bits. The number of consecutive frame failures (lock mode) or consecutive frame success (check mode) are counted until a preset number is reached causing a change in mode. When consecutive successes are being counted in the check mode the occurrence of a failure resets the counter and the system reverts to the search mode. When consecutive failures are being counted in the lock mode the occurrence of a success resets the counter and the count starts over. If the preset number of consecutive failed frames is not reached the system stays in the lock mode. Since the count begins anew after each reset the idea of a recurrent event is applicable, the recurrent event being a run of a preset length.

This approach is more directly applicable to the lock mode where, under reasonable conditions, it is expected that the first failure run of length ρ will not occur until a large number of trials (frames) have passed. The average number of trials until the first failure run of length ρ is therefore a direct indication of the average time in lock for a given bit error rate, q_b, a given number of bits allowed in error in the sync word, E_L, and a given number of consecutive frames allowed in error, $\rho = F_L$.

In the check mode the number of trials until the first run of $r = F_c$ successful frames is of principle interest where the probability is high that the first r trials result in the first run of r consecutive success. That is, in order to complete the acquisition process (search and check), the sync word found in search must be immediately verified by the next r consecutive sync word checks. However, for the MSFTP-1 type system where the check mode is entered immediately upon finding the first sync word (true or false), the average number of trials until the first success run of $r + 1 = F_c + 1$ gives an approximation to the average number of trials required for acquisition.

The assumption of data frame synchronization (not necessarily "true" sync) is necessary to allow each sync word error check to be considered an independent trial. That is, the search mode is assumed completed with a sync word having been found and the following data stream is being examined once each frame period to determine the number of sync bits in error in that frame. The periodic examination (once each frame period) of the sync word results in "success" or "failure" for each sync word under the allowable bits in error criteria chosen. Analysis of the search mode itself is accomplished by different methods described in Appendix I.

The following symbols with their definitions will be used:

$\mathcal{E}$ = An event satisfying certain properties, e.g. $\mathcal{E}$ is a run of 3 consecutive failures in a sequence of trials (3 consecutive bad sync words in a sequence of sync words).

u_n = The probability that the event $\mathcal{E}$ occurs at the nth trial, the sum $u_o + u_1 + u_2 + \ldots$ does not necessarily equal 1 so that the u_j do not necessarily form a probability distribution. By definition $u_o = 1$.

$U(s) = u_o + u_1 s + u_2 s^2 + \ldots$ = The generating function of the probabilities u_j; u_j is the coefficient of the jth power of s and is the probability that $\mathcal{E}$ occurs at the jth trial. The variable s is arbitrarily chosen with $|s| \leq 1$ or $|s| < 1$ to satisfy convergence requirements.

g_n = Probability that the event $\mathcal{E}$ occurs for the first time at the nth trial; the sum $g_o + g_1 + g_2 + \ldots$ equals 1 and the g_k form a probability distribution. By definition $g_o = 0$.

$G(s) = g_o + g_1 s + g_2 s^2 + \ldots$ = The generating function of the probabilities g_k. g_k is the coefficient of the kth power of s and is the probability that $\mathcal{E}$ occurs for the first time at the kth trial. $G(1) = 1$ since the g_k form a probability distribution. The probability symbol g_k will be used when the event is a success run ("good" sync word run) of length r. r is determined by the number of consecutive good sync words ("good" frames) which must occur before the system transfers to the lock mode. This number is selected via a switch. (The symbol $F_c = r$ is used in the discussion of sections 3 and 4).

Since u_n can be written in convolution form as:

$$u_n = g_1 u_{n-1} + g_2 u_{n-2} + g_3 u_{n-3} + \dots g_n u_o, \; u_o = 1,$$

The relation between the generating functions U(s) and and G(s) is, $U(s) - 1 = G(s)\,U(s)$

b_n = probability that the event ε occurs for the first time at the nth trial; the sum $b_o + b_1 + b_2 + ..$ equals 1 and the b_k form a probability distribution. By definition $b_o = 0$

$B(s) = b_o + b_1 s + b_2 s^2 + \dots$ = The generating function of the probabilities b_k and $B(1) = 1$. The probability symbol b_k will be used when the event ε is a failure run ("bad" sync word run) of length ρ. ρ is determined by the number of consecutive "bad" sync words ("bad" frames) which are allowed to occur in the lock mode before the system reverts to the search mode. This number is selected by a switch. (The symbol $F_L = \rho$ is used in the discussion of sections 3 4).

Since u_n can be written in convolution form as:

$$u_n = b_1 u_{n-1} + b_2 u_{n-2} + b_3 u_{n-3} + \dots b_n u_o, \; u_o = 1$$

The relation between the generating functions U(s) and B(s) is, $U(s) - 1 = U(s)\,B(s)$

p = probability of a good sync word. A good sync word contains E_c or fewer bit errors for the check mode and E_L or fewer bit errors for the lock mode. p is associated with "success" or "good" sync word.

q = 1-p = probability of a bad sync word. q is associated with "failure" or "bad" sync word. Each sync word contains 26 bits so that:

$$p_c = \sum_{k=0}^{E_c} \binom{26}{k} q_b^k p_b^{26-k},$$ for the check mode where q_b is the bit error probability $q_b = 10^{-1}, 10^{-2}$ etc. $p_b = 1 - q_b$

and

$$p_L = \sum_{k=0}^{E_L} \binom{26}{k} q_b^k p_b^{26-k},$$ for the lock mode, where in each case the sync bits are assumed to be independent.

E_c is determined by a switch setting, and

E_L is determined by a switch setting

t_n = the probability that ε does not occur in n trials, eg. t_n = probability that no failure run length 3 occurs in n trials.

$$t_n = b_{n+1} + b_{n+2} + b_{n+3} + \ldots$$

The probabilities t_n are for the "tail" of the probability distribution b_n. The probabilities t_n do not form a probability distribution.

$T(s) = t_o + t_1 s + t_2 s^2 + \ldots$ = The generating function of the probabilities t_n

h_n = the probability that ε does not occur in n trials, e.g. h_n = The probability that no success run of length 3 occurs in n trials

$$h_n = g_{n+1} + g_{n+2} + g_{n+3} + \ldots$$

The probabilities h_n are for the "tail" of the probability distribution g_n. The probabilities h_n do not form a probability distribution.

$H(s) = h_o + h_1 s + h_2 s^2 + \ldots$ = The generating function of the probabilities h_n.

Based on partial fraction expansions of the several defined generating functions, approximations for the probabilities will be given as:

$$G(s) = \frac{P(s)}{Q(s)}$$

$g_n \approx \frac{K_g}{x^{n+1}}$ where K_g is a constant determined from substitution of the smallest denominator root, x, into $\frac{-P(s=x)}{Q'(s=x)}$

x is the smallest root of $Q(s)$; $Q' = \frac{dQ(s)}{ds}$

$$H(s) = \frac{A(s)}{Q(s)}$$

$$h_n \approx \frac{K_h}{x^{n+1}} \; ; \; K_h = \frac{-A(s=x)}{Q'(s=x)}$$

$$B(s) = \frac{X(s)}{Y(s)}$$

$b_n \approx \frac{K_b}{y^{n+1}}$, $K_b = \frac{-X(s=y)}{Y'(s=y)}$; for y = the smallest root of $Y(s)$;

$$T(s) = \frac{C(s)}{Y(s)}$$

$$t_n \approx \frac{K_t}{y^{n+1}} \; ; \; K_t = \frac{-C(s=y)}{Y'(s=y)}$$

The relation between the generating function G(s) of the probability distribution g_n and the generating function H(s) for the "tail" probabilities $h_n = g_{n+1} + g_{n+2} + \ldots$ is:

$$H(s) = \frac{1-G(s)}{1-s}$$

Similarly for B(s) and T(s):

$$T(s) = \frac{1-B(s)}{1-s}$$

The formulas for determining the mean and variance from the generating functions are:

The expectation (mean), $u_g = \sum kg_k$

$u_g = G'(1) = H(1)$: for the average trial number at which the first success run of length r is completed.

The variance, $\sigma_g^2 = \sum k^2 g_k - \mu_g^2$

$\sigma_g^2 = G''(1) + G'(1) - G'^2(1) = 2H'(1) + H(1) - H^2(1)$; for the second moment about the mean of the trial number at which the first success run of length r occurs.

The expectation (mean), $\mu_b = \sum_k kb_k$

$\mu_b = B'(1) = T(1)$; for the average trial number at which the first failure run of length ρ is completed.

The variance, $\sigma_b^2 = \sum_k k^2 b_n - \mu_b^2$

$\sigma_b^2 = B''(1) + B'(1) - B'^2(1) = 2T'(1) + T(1) - T^2(1)$; for the second moment about the mean of the trial number at which the first failure run of length ρ occurs.

Some examples of calculating probabilities of recurrent events will now be given using the preceeding definitions.

Section 1: let $\mathcal{E}$ = a success run of length r
u_n = the probability that $\mathcal{E}$ occurs at the nth trial

g_n = the probability that $\mathcal{E}$ occurs for the first time at the nth trial. (After the first occurrence of $\mathcal{E}$ the trials start over and the number of trials until the second occurrence of $\mathcal{E}$ is again a random variable. After the second occurrence of $\mathcal{E}$ the trials start over and the number of trials until the third occurrence of $\mathcal{E}$ is a random variable etc. Hence the name recurrent event).

$$\left.\begin{array}{l} u_o = 1 \\ g_o = 0 \end{array}\right\} \text{by definition}$$

writing u_n as; $u_n = g_1\, u_{n-1} + g_2\, u_{n-2} + g_3\, u_{n-3} + \ldots g_n u_o$

gives the form of the convolution $g_k^* u_{n-k}$

that is:

$$P_r\{\mathcal{E} \text{ occurs at nth trial}\} = P_r\left\{\begin{array}{l}\mathcal{E} \text{ occurs for 1st time at 1st} \\ \text{trial and n-1 trials later}\end{array}\right\}$$

__or__ $P_r\{\mathcal{E}$ occurs for 1st time at 2nd trial __and__ n-2 trials later$\}$

__or__ etc.

the convolution $g_k^* u_{n-k}$ has the generating function $U(s)G(s)$,

using p = probability of success at each trial, $(q = 1-p)$

Equation II-1) $u_n + u_{n-1}\, p + u_{n-2}p^2 + \ldots u_{n-(r-1)}p^{r-1} = p^r$, $r \leq n$ is the probability that the r trials n-(r-1) to n all result in success with $u_o = 1$ by definition and $u_1 = u_2 = u_3 = u_{r-1} = 0$

The generating function $U(s) = u_o + u_1 s + u_2 s^2 + \ldots$ is obtained by multiplying each side of equation 1) by s^n, $r \leq n$ as follows

$$u_r s^r = p^r s^r$$

$$u_{r+1}s^{r+1} + pu_r s^{r+1} = p^r s^{r+1}$$

$$u_{r+2}s^{r+2} + pu_{r+1}s^{r+2} + p^2 u_r s^{r+2} = p^r s^{r+2}$$

$$\vdots$$

etc.

so that

$$U(s)-1 + sp(U(s)-1) + s^2p^2\,(U(s)-1) + \ldots s^{r-1}p^{r-1}\,(U(s)-1) = p^rs^r(1 + s + s^2 + s^3 + \ldots$$

$$U(s)\,(1 + ps + p^2s^2 + \ldots p^{r-1}s^{r-1}) - (1 + ps + p^2s^2 \,..\, p^{r-1}s^{r-1}) = \frac{p^rs^r}{1-s}$$

$$(U(s)-1)\,.\,\frac{1-p^rs^r}{1-ps} = \frac{p^rs^r}{1-s}$$

$$U(s) = 1 + \frac{(1-ps)}{(1-p^rs^r)}\;\frac{p^rs^r}{(1-s)} = \frac{1-s + qp^rs^{r+1}}{(1-s)(1-s^rp^r)}$$

using:

$$G(s) = \frac{U(s)-1}{U(s)}$$

$$G(s) = \frac{\dfrac{(1-ps)p^rs^r}{(1-p^rs^r)(1-s)}}{\dfrac{1-s+qp^rs^{r+1}}{(1-s)(1-s^rp^r}} = \frac{(1-ps)p^rs^r}{1-s+qp^rs^{r+1}}$$

$$\text{and } H(s) = \frac{1-G(s)}{1-s} = \frac{1-s-p^rs^r(1-s)}{(1-s)(1-s+qp^rs^{r+1})} = \frac{1-p^rs^r}{1-s+qp^rs^{r+1}}$$

One method of producing the probabilities g_n is to divide out $G(s)$ by long division. This will be illustrated for the particular case $r = 3$. (A success run of length 3).

$$G_3(s) = p^3s^3+qp^3s^4+qp^3s^5+qp^3s^6+qp^3(1-p^3)s^7+qp^3[1-p^3(1+q)]s^8+qp^3[1-p^3(1+2q]s^9$$
$$+qp^3[1-p^3(1+3q)]s^{10}+q^2p^3\{1-p^3[2+3q-p^3]\}s^{11}+\ldots$$

$$
\begin{array}{r|l}
1-s+qp^3s^4 & p^3s^3-p^4s^4 \\
 & \underline{-p^3s^3+p^3s^4 \qquad -qp^6s^7} \\
 & \quad qp^3s^4 \qquad -qp^6s^7 \\
 & \underline{\quad -qp^3s^4+qp^3s^5 \qquad\qquad} \\
 & \qquad qp^3s^5 \;\; -qp^6s^7 \\
 & \qquad\quad \vdots \\
 & \qquad\quad \text{etc.}
\end{array}
$$

Thus the probability that the first success run of length 3 is completed at the 3rd trial (g_3) is p^3, at the 7th trial (g_7) is $qp^3\ (1-p^3)$, at the 11th trial (g_{11}) is $q^2p^3\{(1-p^3[2+3q-p^3]\}$...etc. The first few terms could be determined by direct reasoning without too much difficulty, higher order terms would be less obvious and a general formula hard to come by. The recursive relation of equation 1) and the resulting G(s) generate the probabilities directly, (although somewhat laborously when done by hand). It seems reasonable that digital computing techniques could be effectively applied if many of the probability coefficients of the generating functions were desired to show the distribution. (See Figure AII-1)

Using $\mu_g = H(1) = \dfrac{1-p^r}{qp^r}$

$\mu_g = \dfrac{1-p^3}{qp^3}$ = the average number of trials until the first success run of length 3 is completed

and

$$\sigma_g^{\,2} = 2H'(1) + H(1) - H^2(1)$$

$$= 2\left(\frac{1}{q^2p^{2r}} - \frac{1}{q^2p^r} - \frac{1}{qp^r} + \frac{1}{q} - \frac{r}{qp^r}\right) + \frac{(1-p^r)}{qp^r} - \frac{(1-p^r)^2}{q^2p^{2r}}$$

$$\sigma_g^{\,2} = \frac{1}{q^2p^{2r}} - \frac{(2r+1)}{qp^r} - \frac{p}{q^2} = \frac{1}{q^2p^6} - \frac{7}{qp^3} - \frac{p}{q^2}\ , \text{ for } r = 3;$$

the variance for the number of trials until the first success run of length 3 is completed.

To illustrate let $p_c = \frac{3}{4}$, $q_c = 1-p_c = \frac{1}{4}$ that is, assume that the probability of a "good" sync word is 3/4 then:

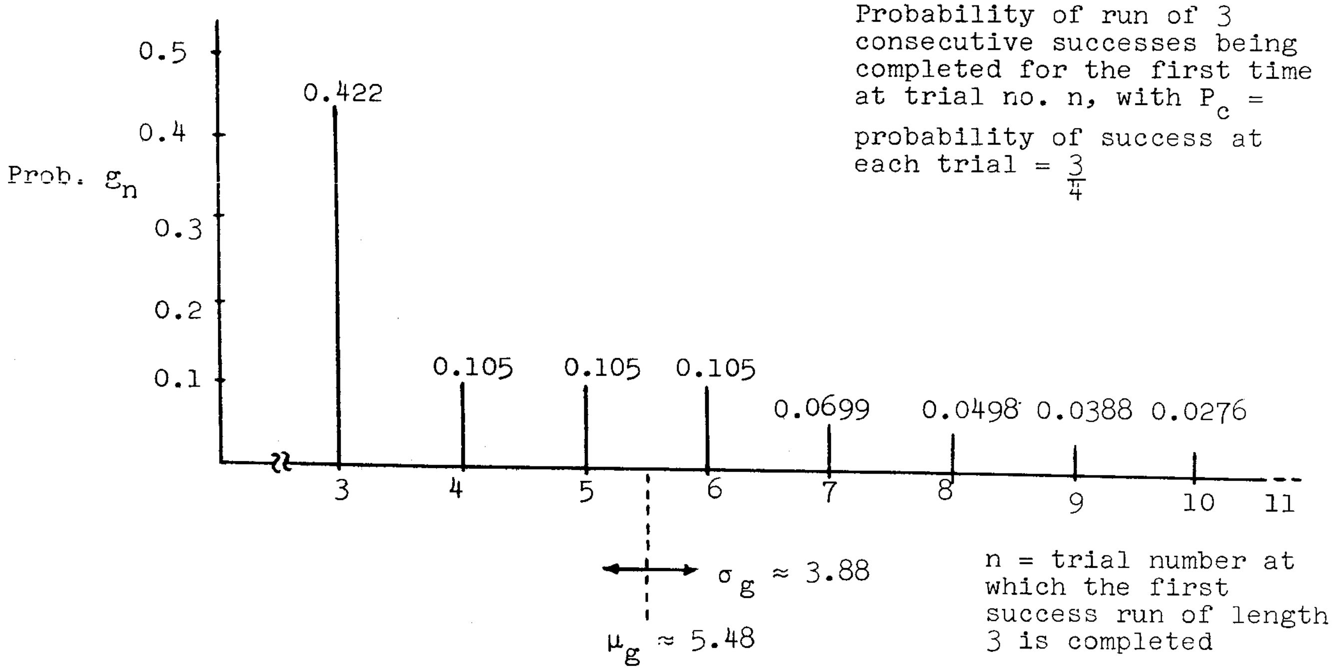
$$\mu_g = \frac{1-(3/4)^3}{(\frac{1}{4})(3/4)^3} \approx 5.48$$

and $\sigma_g^{\,2} = \dfrac{1}{\frac{1}{16}\left(\frac{3}{4}\right)^6} - \dfrac{7}{(\frac{1}{4})(3/4)^3} - \dfrac{3/4}{(\frac{1}{4})^2} \approx 11$

so that $\sigma_g \approx 3.31$

The distribution of the probabilities is plotted for $p_c = 3/4$ in Figure A II-1. These probabilities are the g_n for g_3 through g_{10} obtained by dividing out the generating function $G_3(s)$. The methods of approximating the g_n and the h_n will now be illustrated.

FIG. AII-1 DISTRIBUTION OF PROBABILITIES PLOTTED FOR $p_c = 3/4$



Rounding off μ_g in Figure A II-1 to the integral value $\mu_g = 6$, and σ_g to the integral value $\sigma_g = 3$, the probability that three consecutive "good" frames occur at the 6th frame or before is approximately 0.74 or ≈ 75%. The probability that three consecutive "good" frames occur at the 9th frame $(\mu_g+\sigma_g)$ or before is approximately 0.88 or ≈ 90%. The probability is approximately 0.5 that the first run of three "goods" will occur at the 4th frame or before (3rd plus 4th), but the distribution indicates that it is also fairly probable (approx. 0.08) that more than 10 frames will be required. Thus $h_{10} \approx 0.08$.

By utilizing partial fraction expansions of the generating functions G(s) and H(s) the probabilities g_n and h_n can be approximated by:

$$g_n \approx \frac{(x-1)(1-px)}{(r+1-rx)q} \cdot \frac{1}{x^{n+1}}$$, where x is the smallest root of the denominator of G(s) and H(s).

$G(s) = \frac{P(s)}{Q(s)}$, $Q(s) = 1-s+qp^r s^{r+1}$, and $\frac{(x-1)(1-px)}{(r+1-rx)q} = \frac{-P(s=x)}{Q'(s=x)}$

also, $h_n \approx \frac{1-px}{(r+1-rx)q} \cdot \frac{1}{x^{n+1}}$

again using $p_c = 3/4$ $q_c = 1-p = \frac{1}{4}$, and $r = 3$

solving $x - \frac{1}{4}(3/4)^3 x^4 = 1$ gives $x \approx 1.375$ by trial and error so that
$g_n \approx \frac{0.346}{(1.375)^{n+1}}$;

specifically $g_{10} \approx \frac{0.346}{(1.375)^{11}}$

for this example $g_{10} \approx 0.0105$ by the approximation method

whereas $g_{10} \approx 0.0276$ as obtained from the coefficient of s^{10} in $G_3(s)$

and $h_n \approx \frac{0.924}{(1.375)^{n+1}}$

specifically $h_{10} \approx \frac{0.924}{(1.375)^{11}} \approx 0.03$ by the approximation method

whereas $h_{10} \approx 0.08$ as shown in Figure A II-1.

Again it seems reasonable that digital computing techniques could be effectively applied to estimating the probabilities as well as producing the probability coefficients of the generating functions as previously mentioned.

Section 2:

An identical procedure to that illustrated in Section 1 gives the generating functions B(s) and T(s) and the probabilities b_n and t_n when failure runs of length ρ are considered.

For this case:

$\mathcal{E}$ = a failure run of ρ

u_n = the probability that $\mathcal{E}$ occurs at the nth trial.

b_n = the probability that $\mathcal{E}$ occurs for the first time at the nth trial. (Now the recurrent event is a failure run of length ρ)

This time utilizing the convolution $b_k * u_{n-k}$, and the relation $B(s) = \frac{U(s)-1}{U(s)}$ gives:

$$B(s) = \frac{(1-qs)q^\rho s^\rho}{1-s+pq^\rho s^{\rho+1}},$$ which is identical in form to G(s) but with p and q interchanged and r replaced by ρ.

And similarly since $T(s) = \frac{1-B(s)}{1-s}$

$$T(s) = \frac{1-q^\rho s^\rho}{1-s+pq^\rho s^{\rho+1}}$$

In keeping with the example of section 1 the first few coefficients of B(s) will be shown by long division for $\rho = 3$.

$$B_3(s) = q^3s^3+pq^3s^4+pq^3s^5+pq^3s^6+pq^3(1-q^3)s^7+pq^3[1-q^3(1+p)]s^8+pq^3[1-q^3(1+2p)]s^9+pq^3[1-q^3(1+3p)]s^{10}+\ldots.$$

$$1-s+pq^3s^4 \;\big)\; q^3s^3-q^4s^4$$
$$\underline{-q^3s^3+q^3s^4 \qquad -pq^6s^7}$$
$$pq^3s^4 \qquad -pq^6s^7$$
$$\underline{-pq^3s^4+pq^3s^5}$$
$$\vdots$$
etc.

The coefficients of B(s) are identical in form to those of G(s) with p and q interchanged. There is one important difference however with respect to application. For cases of interest to this study the condition $p_L > q_L$ or $p_L \gg q_L$ will prevail. (Hopefully the probability of a "good" sync word is considerably greater than that for a "bad" sync word). This means that the initial coefficients of B(s) which can be conveniently determined by hand computation as above will be small. (In fact, all of the coefficients will be small). It is of course desirable that the probability of a failure run occurring in relatively few trials be small.

To illustrate again let $p_L = 3/4$, $q_L = \frac{1}{4}$ for $\rho = 3$, a failure run of length 3. For this case $\mu_b = T(1) = \frac{1-q^\rho}{pq^\rho} = \frac{1-(\frac{1}{4})^3}{(3/4)(\frac{1}{4})^3} \approx 84$

and

$$\sigma_b^2 = 2T'(1) + T(1) - T^2(1) = 2\left(\frac{1}{p^2q^{2\rho}} - \frac{1}{p^2q^\rho} - \frac{1}{pq^\rho} + \frac{1}{p} - \frac{\rho}{pq^\rho}\right) + \frac{1-q^\rho}{pq^\rho} - \frac{(1-q^\rho)^2}{p^2q^{2\rho}}$$

$$\sigma_b^2 = \frac{1}{p^2q^{2\rho}} - \frac{(2\rho+1)}{pq^\rho} - \frac{q}{p^2} = \frac{1}{(\frac{3}{4})^2(\frac{1}{4})^6} \cdot \frac{7}{(\frac{3}{4})(\frac{1}{4})^3} - \frac{\frac{1}{4}}{(\frac{3}{4})^2} \approx 6683$$

$$\sigma_b \approx 82$$

The average trial number at which the first run of 3 consecutive failures is completed ($\mu_b \approx 84$) lies far beyond the number which can be conveniently divided out by hand to determine the probability coefficients for the example $B_3(s)$. This emphasizes the usefulness of the "tails" of the distribution, t_n, which give the probabilities that the first failure run of length ρ does not occur until after the nth trial. It also emphasizes the usefulness of the approximation formulas for determining the probabilities b_n and t_n.

Also the potential usefulness of digital computing techniques becomes even more apparent.

The illustration for failure runs of length 3 and $p_L = 3/4$ will now be continued by concentrating on the "tail" probabilities, t_n, and the approximating formulas for t_n and b_n.

First an alternate method of finding the probability that no failure run of length ρ occurs in n trials will be given. This is equivalent to the probability, t_n, that the first failure run of length ρ occurs after the nth trial. The method used is to find a recursive relation for the probabilities so that an iteration process can be used to find the kth probability in terms of the k-1th and k-2th...0th

let ν_n = probability of no failure run of length ρ in n trials

where $\nu_o = 1$, and p = probability of success at each trial, q=1-p

for $\rho = 1$

$$\nu_n = p\,\nu_{n-1}, \quad 1 \leq n, \quad \nu_o = 1$$

$$\nu_1 = p$$

$$\nu_2 = p^2$$

$$\nu_3 = p^3$$

$\vdots$

etc. for this case all trials must result in success for $\rho = 2$

$$\nu_n = p\nu_{n-1} + pq\,\nu_{n-2},\ 2 \leq n,\ \nu_o = \nu_1 = 1$$

$$\nu_2 = p + pq = 1-q^2$$

$$\nu_3 = p(p+pq) + pq$$

$$\nu_4 = p[p(p+pq)+pq]+pq(p+pq]$$

$$\vdots$$

etc.

for $\rho = 3$

$$\nu_n = p\nu_{n-1} + pq\,\nu_{n-2} + q^2p\,\nu_{n-3},\ 3 \leq n,\ \nu_o = \nu_1 = \nu_2 = 1$$

$$\nu_3 = p+pq+q^2p = 1-q^3$$

$$\nu_4 = p(1-q^3)+pq + q^2p = 1-q^3(1+p)$$

$$\nu_5 = p[1-q^3(1+p)]+pq(1-q^3)+q^2p = 1-q^3(1+2p)$$

$$\vdots$$

etc.

The probabilities for $\rho = 3$ are shown in more compact form to allow comparison to the results to be obtained from $T_3(s)$.

for $\rho = 4$

$$\nu_n = p\,\nu_{n-1} + pq\,\nu_{n-2} + pq^2\,\nu_{n-3} + pq^3\,\nu_{n-4},\ 4 \leq n,\ \nu_o = \nu_1 = \nu_2 = \nu_3 = 1$$

for $\rho = 5$

$$\nu_n = p\nu_{n-1} + pq\nu_{n-2} + pq^2\,\nu_{n-3} + pq^3\,\nu_{n-4} + pq^4\,\nu_{n-5},\ 5 \leq n$$

$$\nu_o = \nu_1 = \nu_2 = \nu_3 = \nu_4 = 1$$

$$\vdots$$

etc.

These recursive relations can be used to find the probabilities directly by hand calculation until one becomes tired. The application of digital computing techniques to these recursive formulas is certainly desirable where many probabilities for many variations of the parameters are required.

$$\text{For } \rho = 3; \; T_3(s) = \frac{1-q^3s^3}{1-s+pq^3s^4}.$$

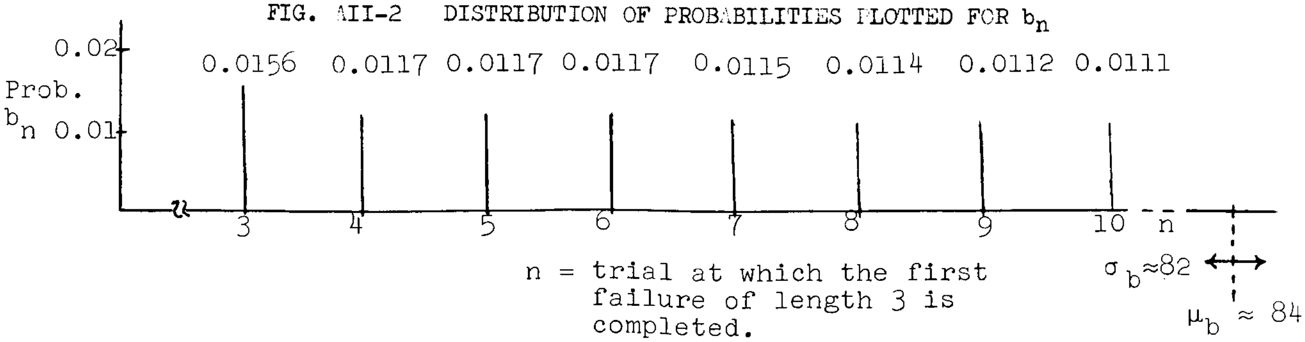
$$T_3(s) = 1+s+s^2+(1-q^3)s^3+[1-q^3(1+p)]s^4+[1-q^3(1+2p)]s^5+[1-q^3(1+3p)]s^6$$
$$+[1-q^3(1+4p)+pq^6]s^7+[1-q^3(1+5p)+pq^6(2+p)]s^8$$
$$+[1-q^3(1+6p)+3pq^6(1+p)]s^9$$
$$+[1-q^3(1+7p)+pq^6(4+3p)]s^{10}+\ldots\ldots$$

$$\begin{array}{r|l}
1-spq^3s^4 & 1-q^3s^3 \\
 & \underline{-1+s \qquad -pq^3s^4} \\
 & s \qquad -q^3s^3 \quad -pq^3s^4 \\
 & \underline{-s+s^2 \qquad\quad -pq^3s^5} \\
 & s^2 \quad -q^3s^3 \quad -pq^3s^4 \quad -pq^3s^5 \\
 & \underline{s^2 \quad + s^3 \qquad\qquad\qquad\qquad} \\
 & \vdots \\
 & \text{etc.}
\end{array}$$

So that $T_3(s) = t_o + t_1s+t_2s^2+\ldots$ has the coefficients $t_n = \nu_n$ for the probabilities that the first failure run of length 3 occurs after the nth trial. The probability, t_2, that the first failure run of length 3 occurs after the 2nd trial is 1, after the 4th trial, $t_4 = [1-q^3(1+p)]$, after the 10th trial, $t_{10} = [1-q^3(1+7p)+pq^6(4+3p)]$ etc. The first few coefficients, t_n, of $T_3(s)$ are seen to be the same as the ν_n obtained from the recursive relation for $\rho = 3$.

Figure AII-2 gives a plot of the first few probabilities, b_n, for the distribution of probabilities that the first failure run of length 3 occurs at the nth trial with $q_L = \frac{1}{4}$.

FIG. AII-2 DISTRIBUTION OF PROBABILITIES PLOTTED FOR b_n



The number of coefficients, b_n, obtained from the generating function $B_3(s)$ are seen to be far short of the mean $\mu_b \approx 84$. The "tail" t_n, of the distribution, b_n, of Figure AII-2 will be more useful. By the approximation method for obtaining the probabilities t_n, that the first failure run of 3 occurs after the nth trial the following estimations are made:

t_{84} = the probability that the first failure run of 3 occurs after μ_b, the average trial at which the first failure run of 3 occurs.

$t_{84} \approx 0.324$, therefore there is approximately a 70% chance that the first failure run of 3 occurs before the average number of trials for that event is reached

for $t_n \approx 0.5$, $n \approx 56$, so that approximately half the time (median) the failure run of 3 will occur after the 56th trial.

for $n = \mu_b + \sigma_b = 166$, $t_n \approx 0.12$, there is about a 90% chance that the failure run of 3 occurs before the $\mu_b+\sigma_b$ trial.

By utilizing partial fraction expansions of the generating functions B(s) and T(s) the probabilities b_n and t_n can be approximated by:

$$b_n \approx \frac{(y-1)(1-qy)}{(\rho+1-\rho y)p} \cdot \frac{1}{y^{n+1}}$$ where y is the smallest root of the denominator of B(s) and T(s)

$$t_n \approx \frac{(1-qy)}{(\rho+1-\rho y)p} \cdot \frac{1}{y^{n+1}}$$

the equation $y-pq^{\rho}y^{\rho+1} = 1$ gives y for $B(s) = \frac{X(s)}{Y(s)}$

$$Y(s) = 1-s+pq^{\rho}s^{\rho+1}$$

and $\frac{(y-1)(1-qy)}{(\rho+1-\rho y)p} = \frac{-X(s=y)}{Y'(s=y)}$

using $p_L = 3/4$, $q_L = 1-p_L = \frac{1}{4}$ and $\rho = 3$

and solving $y-(3/4)(\frac{1}{4})^3y^4 = 1$ gives $y \approx 1.013$ by trial and error

so that $b_n \approx \frac{0.0136}{(1.013)^{n+1}}$; specifically $b_{10} \approx 0.0118$

for this example $b_{10} \approx 0.0118$ by the approximation method whereas $b_{10} \approx 0.0111$ as obtained from the coefficient of s^{10} in B(s)

and $t_n \approx \frac{1.047}{(1.013)^{n+1}}$, specifically $t_{10} \approx \frac{1.047}{(1.013)^{11}} \approx 0.907$

for this example $t_{10} \approx 0.907$ by the approximation method
whereas $t_{10} \approx 0.903$ as obtained from the coefficient of s^{10} in T(s).

An even more simple linear expression for t_n in terms of the average value μ_b can be obtained by additional approximations based on the partial fraction approach. With the restriction that q_L^{ρ} be small so that μ_b is large, $\mu_b = \frac{1-q_L^{\rho}}{p_L q_L^{\rho}} \quad \frac{1}{p_L q_L^{\rho}}$, and that $\rho \leq n \ll \mu_b$ a reasonable approximating formula for t_n is obtained as follows:

For y, the smallest root of the denominator,

$$y \approx 1 + \frac{1}{\mu_b}\left[1 + \frac{\rho+1}{\mu_b} + \frac{(\rho+1)^2}{\mu_b^{\,2}} + \frac{(\rho+1)^3}{\mu_b^{\,3}} + \ldots\right]$$

$$y \approx 1 + \frac{1}{\mu_b}\left[\frac{1}{1-\frac{(\rho+1)}{\mu_b}}\right] = \frac{\mu_b-\rho}{\mu_b-\rho-1}$$

Applying this approximate value of y to the equation,

$$t_n = \frac{(1-q_L y)}{(\rho+1-\rho y)p_L} \cdot \frac{1}{y^{n+1}} \quad \text{gives:}$$

$$t_n \approx \frac{p_L(\mu_b-\rho)-1}{p_L(\mu_b-2\rho-1)} \cdot \left(1 - \frac{1}{\mu_b-\rho}\right)^{n+1}$$

$$t_n \approx \frac{\mu_b+1}{\mu_b-\rho} \cdot \left[1 - \frac{(n+1)}{\mu_b-\rho}\right]$$

for the previous example $q_L = \frac{1}{4}$, $\rho = 3$, $\mu_b \approx 84$

$t_{10} \approx \frac{85}{81} \cdot \left(1 - \frac{11}{81}\right) \approx 0.907$, which agrees with the previous approximation where y was found by trial and error. In this example q_L is not particularly small, but $q_L^{\rho} = (\frac{1}{4})^3$ allows the condition $\mu_b \approx \frac{1}{p_L q_L^{\rho}} = \frac{1}{(\frac{3}{4})(\frac{1}{4})^3} \approx 85$ to agree quite closely with $\mu_b \approx 84$. This approximation method is clearly restricted to $n \ll \mu_b$. All n less than $\mu_{b/10}$ will probably allow reasonable estimates.

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13. ABSTRACT

An analysis has been made of data frame synchronization control for Apollo downlink telemetry. The basic objective was to determine the effects of varying the switch-controlled error tolerance thresholds of two types of existing operational equipment. Criteria for setting these error tolerance switches are presented. The criteria are established to control the average time spent in the acquisition and lock modes of the data decommutation process. Any choice of switch settings would be dependent upon the data quality and quantity requirements of a particular mission. Examples of switch setting criteria are given which depend upon "maximum acceptable" bit error rates based on assumed operational requirements.

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14. KEY WORDS

Synchronization Threshold Control

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Synchronization Criteria